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**ON EQUIVALENCE BETWEEN ISO-STRUCTURES
AND THEIR ORTHODOX COUNTERPARTS**

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Abstract

We explain how every (possibly non-commutative) unital ring $\mathcal{R}$ of Hilbert space operators equipped with the new iso-product $a \star b = aTb$ associated to a given positive-definite element $T \in \mathcal{R}$, $T > 0$ is always isomorphic to the initial ring $\mathcal{R}$. One of the implications of this simple fact is that all the intrinsic constructions with the appropriate iso-structures, are equivalent to their non-iso counterparts. It also invites for an in-depth revision of how physical ideas and concepts expressed in the iso-language could be most effectively mathematically described. In this conceptual framework, we discuss the ring $\ast$ -structures, some ternary structures and the concept of iso-invariant notions. In particular, the collection of all these isomorphic binary iso-structures is equivalent to a single ternary ring structure on $\mathcal{R}$. We also explain, within the example of a standard quantum Euclidean plane, how the iso-products can be used to construct quantized & non-commutative counterparts for the classical structures.

1 Introduction

Iso Structures

An integral part of the program of iso-mathematics [8, 9, 10] is the idea that, in order to naturally understand and express a large array of fundamental phenomena, the products in the relevant algebraic structures should be appropriately modified, extended and generalized. These new products are always constructed in terms of simple algebraic expressions involving the initial products.

In developing the initial and basic examples, it is natural and tempting to consider possible iso variants of the number systems—as the fields of real numbers, complex numbers and the noncommutative quaternions.

As far as the quantum theory is concerned, the most interesting modifications occur when we allow the product in the algebra of quantum observables, to explicitly depend on relevant physical quantities, dynamically reflecting the complete environment of the system.

In this note we prove that the true iso modifications are always necessarily dynamic in their nature. In other words, if we statically change the product in an appropriate ring, with the help of a fixed positive-definite element, the new structure turns out to be isomorphic to the initial one. As such, it can not provide any framework to describe essentially new phenomena.

The situation fundamentally changes if we switch to the dynamical version of the formalism. In this case, the totality of the static structures emerges as a natural geometrical playground for the description of the new product. Therefore we are invited to consider all of these (mutually equivalent) products simultaneously, which naturally leads us to consider the ternary products. A single ternary product includes explicitly all of the static products associated to a given algebraic structure.

Another distinguished manifestation of this dynamical focus is a possibility to interpret diverse interesting quantization algebras, as respective iso-modifications of their classical commutative counterparts.

Preliminaries

Let $\mathcal{R}$ be a ring with zero $0 \in \mathcal{R}$ and unit $1 \in \mathcal{R}$. We shall express the multiplication in $\mathcal{R}$, which we do not require to be commutative, as a simple juxtaposition of things, omitting the multiplication symbol. Thorough this paper we shall assume that $\mathcal{R}$ is realized as a ring of operators acting in a Hilbert space $\mathcal{H}$, in such a way that there is a common everywhere dense subspace $\mathcal{L}$ of $\mathcal{H}$ invariant under all the operators of $\mathcal{R}$, and such that each operator from $\mathcal{R}$ is completely determined by its action in $\mathcal{L}$.

We shall also assume that our ring is closed under taking inverses of positive-definite operators $T \in \mathcal{R}$. The inverse of every such a positive-definite element $T > 0$ is positive-definite, too.

All elements $a \in \mathcal{R}$ such that there exists (necessarily unique) a^{-1} satisfying $aa^{-1} = a^{-1}a = 1$ form a multiplicative group $G(\mathcal{R})$ with the neutral element $1 \in G(\mathcal{R})$ and $(ab)^{-1} = b^{-1}a^{-1}$ for $a, b \in G(\mathcal{R})$.

As far as possible applications in Physics and Mathematics are concerned, we are mainly interested in the situations when $\mathcal{R}$ is an appropriate operator algebra (C*-algebra or von Neumann algebra for instance, if we work with bounded operators). The situations where $\mathcal{R}$ is real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$, or some field extension (like the extension fields of rational functions) all appropriately realized by operators in a Hilbert space, are also quite relevant to us.

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2 The Iso Product

Let us fix a positive-definite element $T \in \mathcal{R}$ and introduce a new product $\star: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ by [8]:

$$a \star b = aTb. \quad (2.1)$$

It is easy to see that $\mathcal{R}$, equipped with this new product replacing the initial one, is a unital ring, the new unit being the inverse T^{-1} . We shall

write $(\mathcal{R}, \star)$ for this new ring, and when referring simply to $\mathcal{R}$ we implicitly assume the initial ring product is there.

Remark. It is worth noticing that groups $G(\mathcal{R})$ and $G(\mathcal{R}, \star)$ coincide. The inverse of $g \in G(\mathcal{R})$ in $(\mathcal{R}, \star)$ is given by $T^{-1}g^{-1}T^{-1}$. Furthermore, as we shall clarify below, the property of positive-definiteness remains the same for both rings.

Remark. The relation between $\mathcal{R}$ and $(\mathcal{R}, \star)$ is symmetric: we can start from $(\mathcal{R}, \star)$ and recover $\mathcal{R}$ by switching $T \rightsquigarrow T^{-2}$, which is the inverse of 1 in $(\mathcal{R}, \star)$ (and symmetrically T is the inverse of the unit of $(\mathcal{R}, \star)$ in $\mathcal{R}$). If we further apply the same type of transform to $(\mathcal{R}, \star)$, obtaining say $(\mathcal{R}, \bullet)$ via another positive-definite $\tilde{T} \in \mathcal{R}$, then $(\mathcal{R}, \bullet)$ is also obtainable directly from $\mathcal{R}$, via $T\tilde{T}T > 0$. Hence, we are dealing with classes of equivalence of multiplicative unital structures.

At a first glance, it might appear that the ring $(\mathcal{R}, \star)$ could happen to be essentially different from $\mathcal{R}$, thus opening doors for some new structures and phenomena, with potentially interesting applications in both Physics and Mathematics.

However this is not the case, as $(\mathcal{R}, \star)$ is always isomorphic to the initial ring $\mathcal{R}$. We shall now describe a natural family of isomorphisms between the two rings. Each of these isomorphisms is associated to a fixed factorization of the form $T = RS$ with $R, S \in G(\mathcal{R})$. Given such a factorization, let us define a map $\Phi: \mathcal{R} \rightarrow \mathcal{R}$ by

$$\Phi(a) = SaR. \quad (2.2)$$

Theorem 1. *Every Φ is a ring isomorphism between $(\mathcal{R}, \star)$ and $\mathcal{R}$: It is bijective, additive and*

$$\Phi(a \star b) = \Phi(a)\Phi(b) \quad (2.3)$$

for all $a, b \in \mathcal{R}$.

Proof. It is clear that Φ is additive and bijective with $\Phi^{-1}(a) = S^{-1}aR^{-1}$. We have $\Phi(a \star b) = S(a \star b)R = S(aTb)R = S(aRSb)R = (SaR)(SbR) = \Phi(a)\Phi(b)$. All this implies that Φ is indeed a ring isomorphism. $\square$

Remark. Notable special cases for R and S are $R = 1, S = T$ and $R = T, S = 1$. The map Φ is then simply left | right translation by T . If R and S commute (so that $RS = SR = T$) then automatically $R, S \in G(\mathcal{R})$ with $R^{-1} = ST^{-1}$ and $S^{-1} = T^{-1}R$. A particularly symmetric solution for Φ occurs when $R = S = \sqrt{T} > 0$. Obviously, to have such a solution T must be ‘square rootable’ in $\mathcal{R}$, a property understandable as an integral part of the purely algebraic positivity notion for $\mathcal{R}$.

Remark. If $\mathcal{R}$ is a C^* -algebra then its every positive-definite element T possesses a unique positive-definite square root $S = \sqrt{T} > 0$.

Remark. If $\mathcal{R}$ is commutative then all the morphisms Φ coincide. In general, the isomorphisms between $(\mathcal{R}, \star)$ and $\mathcal{R}$ can be labeled, via a one-one correspondence, by the automorphisms $\text{Aut}(\mathcal{R})$ of $\mathcal{R}$. Indeed, by fixing Φ all other such isomorphisms are of the form $F\Phi$ where $F \in \text{Aut}(\mathcal{R})$.

3 The $*$ -Structure

Let us now assume that $\mathcal{R}$ is a $*$ -ring in the sense of being closed under taking formally adjoint operators. In other words for every $a \in \mathcal{R}$ there is a (uniquely defined) $a^* \in \mathcal{R}$ such that $\langle \varphi, a\psi \rangle = \langle a^*\varphi, \psi \rangle$ for all $\varphi, \psi \in \mathcal{L}$. This gives a $*$ -structure on $\mathcal{R}$ —which is an involutive, additive and antimultiplicative map $*$: $\mathcal{R} \ni a \mapsto a^* \in \mathcal{R}$.

Remark. Every element $a \in \mathcal{R}$ can be written in a unique way as a sum of one hermitian and one antihermitian element. Explicitly

$$a = \frac{1}{2}(a + a^*) + \frac{1}{2}(a - a^*).$$

Trivial but worth recalling: $0^* = 0$, $1^* = 1$ and $T^* = T$. If $b \in G(\mathcal{R})$ then $b^* \in G(\mathcal{R})$ with $(b^{-1})^* = (b^*)^{-1}$.

Theorem 2. *The $*$ -structure remains antimultiplicative for $\star$, the T -induced product.*

Proof. We have $(a \star b)^* = (aTb)^* = b^*Ta^* = b^* \star a^*$ for every $a, b \in \mathcal{R}$. $\square$

Theorem 3. *If $S = \sqrt{T}$ is in $\mathcal{R}$ then the associated isomorphism Φ is $\star$ -preserving.*

Proof. For in this case $\Phi(a)^* = (SaS)^* = Sa^*S = \Phi(a^*)$, for all $a \in \mathcal{R}$. $\square$

Remark. As we have already remarked, the notion of positivity is the same for $(\mathcal{R}, \star)$ and $\mathcal{R}$. Indeed if we compose the isomorphism Φ with the initial embedding of $\mathcal{R}$ into operators in $\mathcal{L}$, we obtain a realization of $(\mathcal{R}, \star)$ by operators in $\mathcal{L}$. The operators a and $\sqrt{T}a\sqrt{T}$ are always simultaneously positive or not.

Remark. Note that the sole assumption of invertibility of a hermitian T implies the admission of negative-definite values of T in which case a natural anti-isomorphism from $\mathcal{R}$ to $(\mathcal{R}, \star)$ appears, which is equivalent to charge conjugation [11].

Remark. If we are within C^* -algebras, then positive-definite elements $T \in \mathcal{R}$ possess unique positive-definite square roots $S = \sqrt{T} > 0$, as we have already mentioned. Taking into account that the norm in a C^* -algebra is uniquely determined by the $\star$ -algebra structure, we conclude that $(\mathcal{R}, \star)$ is naturally a C^* -algebra too, isomorphic to $\mathcal{R}$.

Remark. It is worth recalling the spectral radius formula $\|a\| = \sqrt{\rho(a^*a)}$ for any C^* -norm, where ρ is the spectral radius operator. Accordingly, the unique C^* -norm on $(\mathcal{R}, \star)$ is given by $\|a\|_\star = \|\Phi(a)\|$, so that the isomorphism Φ is isometric, as it should be.

4 Extended Structures

A qualitatively different situation arises, if we allow the element T to be *external* to $\mathcal{R}$, so now T together with its inverse and $\mathcal{R}$ generate a larger ring, say $\tilde{\mathcal{R}}$. In this case the subring of $\tilde{\mathcal{R}}$ generated by initial generators of $\mathcal{R}$ but using the new product $\star$, can be quite different.

To illustrate this, we shall explicitly construct the Manin quantum plane algebra as an iso-extension of the classical complex Euclidean plane.

Let us consider the complex coordinate z and its conjugate $\bar{z}$, understood as commuting multiplication operators in the algebra of polynomials $\mathcal{L}$ over z and $\bar{z}$. The space $\mathcal{L}$ is equipped with an appropriate scalar product, so that z and $\bar{z}$ are mutually formally adjoint, and we shall assume that

$$\left[\frac{\partial}{\partial z}\right]^* = -\frac{\partial}{\partial \bar{z}}.$$

Let us fix a scalar $q > 0$. We are looking for a positive-definite operator T in $\mathcal{L}$, which induces the new iso-product $\star$ such that

$$z^* \star z = q z \star z^*. \quad (4.1)$$

Here we have denoted the operator representation of $\bar{z}$ by z^* . The above relation reads, in terms of the standard composition of operators in $\mathcal{L}$ as

$$z^* T z = q z T z^*. \quad (4.2)$$

In polar coordinates $r = |z|, \varphi = \arg(z)$ the basic differentiations are

$$\frac{\partial}{\partial z} = \frac{e^{-i\varphi}}{2} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \varphi} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{e^{i\varphi}}{2} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \varphi} \right).$$

There is a natural decomposition of the polynomial space $\mathcal{L}$, particularly tuned to the polar coordinates, into a sum of ‘circular harmonic’ subspaces

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n \quad (4.3)$$

where $\mathcal{L}_n$ are subspaces spanned by monomials $z^k \bar{z}^l$ with $k - l = n$. These are the polynomials ψ satisfying $-i(\partial/\partial \varphi)\psi = n\psi$. In other words, the polynomial eigenfunctions for

$$\frac{\partial}{\partial \varphi} = i \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

Clearly $\mathcal{L}_n \mathcal{L}_m \subseteq \mathcal{L}_{n+m}$ and in particular $z: \mathcal{L}_n \rightarrow \mathcal{L}_{n+1}$ and $z^*: \mathcal{L}_n \rightarrow \mathcal{L}_{n-1}$.

As we shall now see, there exists a natural class of solutions where the operator T reduces in all subspaces $\mathcal{L}_n$ to simple multiplicative operators

$$\mathcal{L}_n \ni \psi \mapsto t_n \psi \in \mathcal{L}_n \quad (4.4)$$

where $t_n \in \mathcal{L}_0$ are (in general) functions of q and $\varrho = r^2 = \bar{z}z$.

In terms of the functions t_n the structural equation (4.2) becomes a recursive system

$$t_{n+1} = qt_{n-1} \quad n \in \mathbb{Z}. \quad (4.5)$$

The simplest solution is given by the sequence of scalars $t_n = q^{n/2}$ where $n \in \mathbb{Z}$. In the operator form we can rewrite this solution as

$$T = \exp\left\{\frac{1}{2}\log(q)\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right\} \quad (4.6)$$

5 Concluding Thoughts

5.1 Iso-Structures Versus Orthodox Counterparts

It follows that the systems of isoreal, isocomplex, isoquaternionic numbers, as well as the operator algebras equipped with the new iso-products, are all isomorphic to their orthodox non-iso counterparts. In particular, all their performances turn out to be, although perhaps a little masqueraded, the same charming things of Old.

However, as explained in the previous section, the situation essentially changes in the dynamic context, where the product defining element is external to the initial algebraic structure, which in this case can be thought of being appropriately refined | contextually modified. The standard quantization is interpretable as one of such possible modifications.

And in resonance with this, it is worth comparing the iso-product expression with the Weyl-Moyal product in the complex Euclidean plane

$$f \star g = f \exp\left\{\hbar\left(\overleftarrow{\frac{\partial}{\partial \bar{z}}}\overrightarrow{\frac{\partial}{\partial z}} - \overleftarrow{\frac{\partial}{\partial z}}\overrightarrow{\frac{\partial}{\partial \bar{z}}}\right)\right\}g$$

and observe an interesting parallelism, both symbolic and visual.

Our iso-quantization method can be naturally extended to quantum domains whose non-commutative algebras are of the form $z^*z = f(zz^*)$ where f is an appropriate analytic function—for example satisfying $f(0) = 0$ and

positive and bijective on positive real numbers. The recursive formulae for the sequence t_n are

$$t_{n+1} = \frac{f[\varrho t_{n-1} t_n]}{\varrho t_n} \quad t_{n-1} = \frac{f^{-1}[\varrho t_n t_{n+1}]}{\varrho t_n} \quad (5.1)$$

where $n \in \mathbb{Z}$. In this more general context, the space $\mathcal{L}$ should be appropriately refined, in order to accommodate for non-polynomial expressions involving ϱ . And if we put $f(w) = qw$ where $q > 0$, this yields us back to our basic example of the Manin quantum plane.

5.2 Physical Ideas & Concepts

As far as the physical applications are concerned, the physics behind $(\mathcal{R}, \star)$ appears to be exactly the same as the physics behind the original $\mathcal{R}$. We are assuming here that $\mathcal{R}$ is a C^* -algebra interpretable as generated by physical observables, and that we are in the framework of Haag-Kastler-Araki Algebraic Physics, where the ultimate goal is to express all of physics intrinsically in terms of C^* -algebras of observables. Including quite subtle and interesting things, like equilibrium states and KMS-formalism, wave functions and representations in Hilbert spaces, phase transitions of Quantum Statistical Mechanics and superselection sectors of Quantum Field Theory [1, 6].

In resonance with the above, it appears that it could be quite interesting and useful, to mathematically reformulate and translate the variety of physical ideas and concepts originally developed in the framework of iso-mathematics, to their equivalent ‘orthodox’ counterparts.

The translated variants would possibly tend to be simpler and more compact, and possibly revealing some important previously hidden underlying structure.

5.3 Multiple Binary Products, Invariants & Ternary Structures

An interesting way to accommodate simultaneously all the iso-products for $\mathcal{R}$, is to consider the following *ternary* operation $\mathcal{R} \times G(\mathcal{R}) \times \mathcal{R} \rightarrow \mathcal{R}$ defined by

$$(x, g, y) \rightsquigarrow xg^{-1}y.$$

This operation is a natural invariant—it does not change if we switch from $\mathcal{R}$ to an arbitrary $(\mathcal{R}, \star)$. By fixing the middle argument g to T^{-1} and allowing the first and the third to vary, we thus recover all the binary iso-products. Furthermore, all the left | right translation maps by the elements of $G(\mathcal{R})$ are symmetries of this unique ternary structure.

In this context, it is natural to consider notions invariant under switching from one product to another. We can call such things iso-invariants. One such a notion is a generalized derivation, a linear map $D: \mathcal{R} \rightarrow \mathcal{R}$ satisfying

$$D(fg) = D(f)g + fD(g) - fD(1)g \quad (5.2)$$

for all $f, g \in \mathcal{R}$, which is equivalent to saying that the operator $D - D(1)$ satisfies the standard Leibniz rule. Indeed, the formula

$$D(T^{-1}) = T^{-1}D(1) + D(1)T^{-1} - T^{-1}D(T)T^{-1} \quad (5.3)$$

ensures that the same defining equation would hold in all T -induced products. From the quantum-geometric perspective [2, 7, 14, 15], the identity (5.2) generalizes the concept of the first order differential operator, which naturally invites us to consider the algebra $\text{Diff}(\mathcal{R})$ of linear transformations in $\mathcal{R}$ generated by all the maps D . The classical analogue of this is the full algebra of differential operators for the underlying space (for example $\mathcal{R} = C^\infty(M)$ for a given compact smooth manifold M). So we can say that $\text{Diff}(\mathcal{R})$ is one important iso-invariant in $\mathcal{R}$.

We can also forget altogether about the addition operation, and consider simply the unital semigroups, with the corresponding multitude of binary multiplication operations morphing into a single ternary operation. Everything further simplifies for classical groups G as the domain of the ternary operation is just $G \times G \times G$. This promotes an explicit view of homogeneity of G , as any element can assume the role of the neutral element.

Interestingly, when naturally generalized to the quantum-geometric level this ternary interpretation of groups brings a whole array of new examples of quantum groups, far beyond the standard framework of Hopf algebras [3]. For example, the quantum 2-torus [13] and the quantum Euclidean plane [5] based on the Heisenberg-Weyl algebra, are not at all equippable with a Hopf algebra structure. This is because there are no characters, they

are ‘pointless’ as quantum spaces. On the other hand the counit of every Hopf algebra is always a character. Luckily, both spaces can be naturally viewed as quantum groups in this ternary, apparently more democratical and participative, world. An important part of the quantum geometrical interpretation for the iso framework, is that the new products form a quantum space, which might be completely ‘pointless’ (in which case we can not pick up a single product in this quantum structure, we can only talk in terms of a quantum collection of, classically non-existent, products).

The ternary structures provide a framework for dynamical iso-products, realizing the idea of contextual dependence of the basic algebraic structure. This could be especially interesting for the physical situations involving the mechanics of non point-like particles [12].

Another important framework for ternary structures is given by Hilbert modules [4] (with Hilbert spaces being a special case). For them, the ternary product is

$$(\psi, \varphi, \xi) \rightsquigarrow |\psi\rangle\langle\varphi|\xi\rangle$$

which in one single expression unifies the Dirac ket-bra diads and bra-kets giving a base for an elegant treatment of Morita equivalent C^* -algebras, where the Morita equivalence between A and B is geometrically realized via the appropriate Hilbert A – B bimodules.

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ON CONCEPT OF Γ -SEMIGROUPS

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Abstract

This paper presents some fundamental results tacit in [20] precisely on regular Γ -semigroups and operator semigroups of a Γ -semigroup, and an overview of some classes of Γ -semigroups yet to be surveyed.

Keywords: Γ -semigroup; Regular Γ -semigroup; Operator semigroups; Semisimple Γ -semigroup; Simple Γ -semigroup and Γ -group.

1 Introduction

The notion of Γ in algebra was first introduced by Nobusawa [14] as a generalization of ring in the form of Γ -ring. Let M and Γ be additive groups such that for all $a, b, c \in M$ and $\gamma, \beta, \alpha \in \Gamma$, we have $a\gamma b \in M$ and $\gamma a \beta \in \Gamma$ for every a, b, γ and β , then M is called a Γ -ring if the following conditions are satisfied:

- (i) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b$,
 $a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$,
 $a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2$,
- (ii) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$,
- (iii) if $a\gamma b = 0$ for any $a, b \in M$, then $\gamma = 0$.

The structure of Γ -rings as initiated by [14] was studied by Barnes [1], Luh [12], Ravishankar and Shukla [15], Buys and Groenewald [4], Booth [2] and Kyuno [?]. Motivated by this generalization of a ring in 1981, Sen [18] defined the concept of a Γ -semigroup as follows:

Let S and Γ be two non-empty sets. Then S is called Γ -semigroup if there exist mappings $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longmapsto a\alpha b$, and $\Gamma \times S \times \Gamma \longrightarrow \Gamma \mid (\alpha, a, \beta) \longmapsto \alpha a \beta$, satisfying the identities $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, as a generalization of a semigroup. This is regarded as two sided Γ -semigroup.

In 1986, Sen and Saha [19] weakened slightly the defining conditions of Γ -semigroup and redefined Γ -semigroup as follows:

Definition 1.1 *Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two non-empty sets. Then S is called a Γ -semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longmapsto a\alpha b \in S$ satisfying $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. The modified definition (regarded as one-sided) of Sen and Saha [19] seems very convenient to work with because most research papers in literature, particularly the notions of idempotent element in a Γ -semigroup and regularity of a Γ -semigroups adopted their concept.*

Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S . Let us extend the given binary relation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1 = a$ for all a in S . It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1 . Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a Γ -semigroup where $\Gamma = \{1\}$. Thus, a semigroup can be considered to be a Γ -semigroup. Also, if $\Gamma = S$, it is obvious that every semigroup is a Γ -semigroup.

Let S be a Γ -semigroup and α be a fixed element of Γ . Define $a \cdot b = a\alpha b$ for all $a, b \in S$. It can be shown that $(S, \cdot)$ is a semigroup and denote this by semigroup by S .

Changphas [5] studied power Γ -semigroups. In [6], Changphas and Kummoon introduced the notions of left and right bases of a Γ -semigroup.

The development of Γ -semigroups hinges on the fact that subsets of a semigroup naturally inherits associativity but not necessarily closed. As a result of this, various generalizations and analogues of corresponding results in semigroup theory have been obtained based on the modified definition. Although Sen and Chattopadhyay [20] had reviewed numerous published papers in varied aspects on this topic, however the present survey focuses on some fundamental results precisely on regular Γ -semigroups, operator semigroups of a Γ -semigroup, and a synopsis of some classes of Γ -semigroups yet to be reviewed.

In the following, some examples of Γ -semigroups are presented.

Example 1.1 Let $S = \mathbb{Z}$ be the set of all integers and $\Gamma = \{n \mid n \in \mathbb{N}\}$. Then S is a Γ -semigroup with the operation defined by $a\alpha b = a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$.

Example 1.2 Let $S = \{-i, 0, i\}$ and $\Gamma = S$. Then S is a Γ -semigroup under multiplication over complex numbers while S is not a semigroup under complex numbers multiplication.

Example 1.3 Let S be a set of all negative rational numbers. Obviously, S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} \mid$

p is prime}. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now, if $a\alpha b$ is equal to the usual product of rational numbers a, α, b ; then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence, S is a Γ -semigroup.

2 Regular Γ -Semigroup

In this section, we present some results on regular Γ -semigroups characterized by one-sided Γ -ideals and also the connection between idempotency and regularity in Γ -semigroups tacit in [20].

The notion of regularity of a Γ -semigroup was introduced in [19] by Sen and Saha. In [19], the authors characterized regularity condition with the help of Γ -ideals of Γ -semigroup. Hila [9] described the regularity of a Γ -semigroups in terms of quasi- and bi-ideals. Braja [3] described the characterization of regular Γ -semigroups using quasi-ideals. In continuation with this concept, Xhilari and Braja [21] studied completely regular Γ -semigroups by quasi-ideals and established a necessary and sufficient condition for an element to be completely regular.

Definition 2.1 [19] Let S be a Γ -semigroup. An element $a \in S$ is said to be regular if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. The Γ -semigroup S is said to be a regular Γ -semigroup if every element of S is regular.

Example 2.1 Consider the Γ -semigroup S in Example 1.1. Since for $a \in S$ and if $\alpha = m$ and $\beta = n$ for some $m, n \in \Gamma$, then there exists $x = -m - n - a \in S$ such that $a\alpha x\beta a = a + m + x + n + a = a + m + (-m - n - a) + n + a = a$. Hence, S is a regular Γ -semigroup.

Example 2.2 [20] Let S be the set of all positive integers of the form $4n+1$ and Γ be the set of all positive integers of the form $4n+3$. If $a\alpha b$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$, then S is a Γ -semigroup. Since for the element $2 \in S$, there do not exist any $a \in S$ and $\alpha, \beta \in \Gamma$ such that $2 + \alpha + a + \beta + 2 = 2$, 2 is not a regular element in S . Hence, S is not a regular Γ -semigroup.

Let S be a Γ -semigroup and α be a fixed element of Γ . It is evident that for a Γ -semigroup S , if S_α is a regular semigroup for some $\alpha \in \Gamma$ then S is a regular Γ -semigroup but

(i) S_α may not be a regular semigroup for some $\alpha \in \Gamma$ but S may be a regular Γ -semigroup.

Example 2.3 [20] Let $S = \{(a, 0) : a \in R\} \cup \{(0, b) : b \in R\}$ where R denote the field of real numbers. Let $\Gamma = \{(0, 5), (0, 1), (3, 0), (1, 0)\}$. Define $S \times \Gamma \times S \rightarrow S$ by $(a, b)(\alpha, \beta)(c, d) = (a\alpha c, b\beta d)$ for all $(a, b), (c, d) \in S$ and $(\alpha, \beta) \in \Gamma$, we can show that S is a Γ -semigroup. S_α is not a regular semigroup for any $\alpha \in \Gamma$. Let $(a, 0) \in S$. If $a = 0$, then $(a, 0)$ is regular. Suppose $a \neq 0$, then $(a, 0)(3, 0)(\frac{1}{3a}, 0)(1, 0)(a, 0) = (a, 0)$. Similarly we can show that $(0, b)$ is also regular for all $b \in R$. Hence S is a regular Γ -semigroup.

(ii) In a Γ -semigroup S , if S_α is a regular semigroup for some $\alpha \in \Gamma$, then there may exist $\beta \in \Gamma$ such that S_β is not a regular semigroup.

Example 2.4 [20] Let $S = \{(a, b) : a, b \in R, \text{ the field of real numbers}\}$ and $\Gamma = \{(9, 7), (0, 3)\}$. Defining $(a, b)(\alpha, \beta)(c, d)$ by $(a, b)(\alpha, \beta)(c, d) = (a\alpha c, b\beta d)$ for $(a, b), (c, d) \in S$ and $(\alpha, \beta) \in \Gamma$ we find that S is a Γ -semigroup. In this case $S_{(0,3)}$ cannot be a regular semigroup but $S_{(9,7)}$ is a regular semigroup.

In the theorems given below, the regular Γ -semigroup was characterized by condition on one sided Γ -ideals.

Theorem 2.1 [19] Every Γ -ideal of a regular Γ -semigroup S is a regular Γ -ideal of S .

Theorem 2.2 [19] A Γ -semigroup S is regular if and only if for any left Γ -ideal A and for any right Γ -ideal B of S , $A \cap B = B\Gamma A$.

Corollary 2.1 [19] Let S be a commutative Γ -semigroups. Then S is regular if and only if $A = A\Gamma A$ for all Γ -ideals of S .

Theorem 2.3 [16] Let S be a Γ -semigroup. If $a = a\alpha b\beta a$, $b \in S$ and $\alpha, \beta \in \Gamma$, be a regular element of S , then $\langle a \rangle_r = a\alpha S$ and $\langle a \rangle_l = S\beta a$.

Idempotent elements play an important role in semigroup theory. The notion of idempotent element in a Γ -semigroup was defined by Sen and Saha [19] as follows:

Definition 2.2 *Let S be a Γ -semigroup and $\alpha \in S$. Then $e \in S$ is said to be an α -idempotent if $e\alpha e = e$. The set of all α -idempotents is denoted by E_α . We denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S . If $S = E(S)$, then S is called an idempotent Γ -semigroup.*

The following results show the connection between the idempotency and regularity of Γ -semigroups.

Theorem 2.4 [16] *Let S be a Γ -semigroup. An element $a \in S$ is regular if and only if $\langle a \rangle_r = e\beta S$ for some idempotent $e = e\beta e \in S$ where $\beta \in \Gamma$.*

Theorem 2.5 [16] *Every α -idempotent element in a Γ -semigroup is regular.*

Theorem 2.6 [16] *If a Γ -semigroup S is a regular, then every principal Γ -ideal is generated by a Γ -idempotent.*

3 Operator Semigroups of a Γ -Semigroup

After the introduction of Γ -ring by Nobusawa [14], the notion of operator rings of a Γ -ring was introduced by Kyuno [11]. Booth [2] also studied operator rings to be a very effective tool in studying Γ -ring. In [7], Dutta and Adhikari introduced this notion in both sided Γ -semigroup and studied several results relating this structure.

Definition 3.1 [7] *Let S be a both sided Γ -semigroup and ρ be a relation on $S \times \Gamma$ defined as follows:*

$(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation on $S \times \Gamma$. Let $[x, \alpha]$ denote the equivalence class containing (x, α) . Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to multiplication defined by $[x, \alpha][y, \beta] =$

$[x\alpha y, \beta]$. The semigroup L is called the left operator semigroup of S . Similarly define a relation ξ on $\Gamma \times S$ as follows: $(\alpha, a)\xi(\beta, b)$ if and only if $s\alpha a = s\beta a$ for all $s \in S$ and $\alpha\alpha\gamma = \beta b\gamma$ for all $\gamma \in \Gamma$. Then ξ is an equivalence relation on $\Gamma \times S$. Let $[\alpha, a]$ denote the equivalence class containing (α, a) and $R = \{[\alpha, x] : \alpha \in \Gamma, x \in S\}$. Define a multiplication on R by $[\alpha, x][\beta, y] = [\alpha, x\beta y]$. Then R is a semigroup and it is called the right operator semigroup of S .

In relation to the concept, Dutta and Chattopadhyay [8] slightly changed the defining conditions of Γ -semigroup by Sen and Saha [19] and defined operator semigroups as follows:

Let S be a Γ -semigroup. We define a relation ρ on $S \times \Gamma$ as follows:

$$(x, \alpha)\rho(y, \beta) \Leftrightarrow x\alpha s = y\beta s, \forall s \in S.$$

Obviously ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α) . Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. The semigroup L is called the left operator semigroup of S . The right operator semigroup R of a Γ -semigroup can be defined in a similar manner.

The following definitions and results are analogous to [7].

Definition 3.2 [8] Let S be a Γ -semigroup. Let L and R be the left and right operator semigroups of the Γ -semigroup S . If there exist an element $[e, \delta] \in L$ ($[\delta, e] \in R$) such that $e\delta x = x(x\delta e = x)$ for all $x \in S$, then S is said to have the left unity $[e, \delta]$ (right unity $[\delta, e]$).

Example 3.1 [8] Let S be the set of all integers of the form $4n + 1$ and Γ be the set of all integers of the form $4n + 3$ where n is an integer. If $a\alpha b$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$, then S is a Γ -semigroup. $[1, -1]$ is a left unity of S and $[-1, 1]$ is a right unity of S .

Theorem 3.1 [8] Let S be a Γ -semigroup. If $[e, \delta]$ is left unity of S , then $[e, \delta]$ is the identity element of L .

Theorem 3.2 [8] *Let S be a Γ -semigroup. If $[\mu, f]$ is right unity of S , then $[\mu, f]$ is the identity element of R .*

The next set of theorems (analogous to [8]) show a relation between the set of all ideals of S and that of the left (right) operator semigroup $L(R)$.

Definition 3.3 [8] *Let S be a Γ -semigroup with the left operator semigroup L and the right operator semigroup R . For $P \subseteq L$, $A \subseteq R$ and $Q \subseteq S$ we define*

$$\begin{aligned} P^+ &= \{x \in S : [x, \alpha] \in P \text{ for all } \alpha \in \Gamma\} \text{ and} \\ Q^{+'} &= \{[x, \alpha] \in L : x\alpha s \in Q \text{ for all } s \in S\} \\ A^* &= \{x \in S : [\alpha, x] \in A \text{ for all } \alpha \in \Gamma\} \text{ and} \\ Q^* &= \{[\alpha, x] \in R : s\alpha x \in Q \text{ for all } s \in S\}. \end{aligned}$$

Theorem 3.3 [8] *Let S be a Γ -semigroup with left and right unities and L be its left operator semigroup.*

- (i) *If Q is an ideal of S , then $Q^{+'}$ is an ideal of L .*
- (ii) *If P is an ideal of L , then P^+ is an ideal of S .*
- (iii) *$(Q^{+'})^+ = Q$ and $(P^+)^{+'} = P$.*

Theorem 3.4 [8] *Let S be a Γ -semigroup with left and right unities and R be its right operator semigroup.*

- (i) *If Q is an ideal of S , then Q^* is a right ideal of R .*
- (ii) *If P is a an deal of R , then P^* is a Γ -ideal of S .*
- (iii) *$(Q^*)^* = Q$ and $(P^*)^{+'} = P$.*

Theorem 3.5 [8] *Let S be a Γ -semigroup with left and right unities and let left operator L and R be its left operator semigroup and right operator semigroup respectively. Then there exists an inclusion preserving bijection between the set of all ideals of S and the set of all ideals of $L(R)$ via the mapping $Q \longrightarrow Q^{+'}$ ($Q \longrightarrow Q^*$) where Q is a Γ -ideal of S .*

Dutta and Chattopadhyay [8] also initiated the notions of uniformly strongly prime semigroup, uniformly strongly prime ideal, Rees congruence on Γ -semigroup and uniformly strongly prime Γ -semigroup and studied these via operator semigroups of Γ -semigroup. In [17], Sardar et al. showed that the left operator and right operator semigroups of a Γ -semigroup with unities are Morita equivalence monoid and further established that there is a close connection between the Morita equivalence of monoids and Γ -semigroups.

4 Semisimple Γ -semigroup

The concept of semisimple Γ -semigroup was introduced by Madhusudhana *et al.* [13]. The authors proved that in a semisimple Γ -semigroup S the conditions every ideal of S is prime, S is a primary Γ -semigroup, S is a left primary Γ -semigroup, S is a right primary Γ -semigroup, S is a semiprimary Γ -semigroup, Prime Γ -ideals of S form a chain, Γ -ideals of S form a chain, Principal Γ -ideals of S form a chain are equivalent. The definition of semisimple Γ -semigroup was introduced as follows:

Definition 4.1 [13] *An element a of a Γ -semigroup S is called semisimple if $a \in \langle a \rangle \Gamma \langle a \rangle$, i.e., $\langle a \rangle \Gamma \langle a \rangle = \langle a \rangle$. A Γ -semigroup S is said to be semisimple if every element of S is semisimple.*

Theorem 4.1 [13] *If a is a left (right) regular element of a Γ -semigroup S , then a is semisimple.*

Theorem 4.2 [13] *If a is a regular element of a Γ -semigroup S , then a is semisimple.*

Theorem 4.3 [13] *If a is a completely regular element of a Γ -semigroup S , then a is regular and semisimple.*

Theorem 4.4 [13] *In a semisimple Γ -semigroup S , the following are equivalent.*

- (i) *Every Γ -ideal of S is a prime Γ -ideal.*

- (ii) S is a primary Γ -semigroup.
- (iii) S is a left primary Γ -semigroup.
- (iv) S is a right primary Γ -semigroup.
- (v) S is a semiprimary Γ -semigroup.
- (vi) Prime Γ -ideals of S form a chain.
- (vii) Γ -ideals of S form a chain.
- (viii) Principal Γ -ideals of S form a chain.

We refer the readers to [13] for more details.

5 Simple Γ -semigroup and Γ -group

Definition 5.1 [19] A Γ -semigroup S is called a left (right) simple if it contains no proper left (right) ideal i.e., for every $a \in S$, $S\Gamma a = S$ ($a\Gamma S = S$). A Γ -semigroup S is said to be simple if it has no proper ideal.

Theorem 5.1 [19] A Γ -semigroup S is simple if and only if $S\Gamma a\Gamma S = S$ for every $a \in S$.

Theorem 5.2 [19] S_α is a group if and only if S is both left simple and right simple Γ -semigroup.

Corollary 5.1 [19] Let S be a Γ -semigroup. If S_α is a group for some $\alpha \in \Gamma$, then S_α is a group for all $\alpha \in \Gamma$.

Definition 5.2 [19] A Γ -semigroup S is called Γ -group if S_α is a group for some (hence for all) $\alpha \in \Gamma$.

Example 5.1 Let S be the set of all integers of the form $4n+1$ and Γ be the set of all integers of the form $4n+3$. Define $a\alpha b = a + \alpha + b$ for $a, b \in S$ and $\alpha \in \Gamma$. We can show that S is a Γ -semigroup. Now $3 \in \Gamma$. We consider the system S_3 . Notice that $-3 \in S$ and $-3 \cdot (4n+1) = -3+3+(4n+1) = 4n+1$. Also, $(4(-2-n)+1) \cdot (4n+1) = (4(-2-n)+1)+3+4n+1 = -3$. Hence, -3 is a left identity in S_3 and for each $4n+1 \in S_3$ there exists a left inverse namely $4(-2-n)+1$ in S_3 . This shows that S_3 is a group and hence S is a Γ -group.

Example 5.2 Let A and B be two non-empty sets. Let S denote the set of bijective mappings from A to B and Γ the set of bijective mappings from B to A . Suppose Γ is non-empty. Then S is non-empty. Let $f, g \in S$ and $\alpha \in \Gamma$. Then $f\alpha g$, the usual mapping product, is a bijective mapping from A to B . Hence, $f\alpha g \in S$. We can show that S is a Γ -group.

Definition 5.3 [19] Let S be a Γ -semigroup. An element x of S is called a left (right) zero element if $x\alpha a = x$ ($a\alpha x = x$) for all $a \in S$ and for all $\alpha \in \Gamma$. An element x of S is called a zero element of S if it is both a left zero and a right zero element of S . A Γ -semigroup S may not contain a zero element, but if it contains a zero element then it is unique. The zero element of a Γ -semigroup is denoted by 0 .

Definition 5.4 [19] A Γ -semigroup S with zero is said to be a 0 - Γ -group if $S \setminus \{0\}$ is a Γ -group.

Example 5.3 Let $S = Q \times Q$, where Q is the multiplicative group of rational numbers and let $\Gamma = Z \times Z$, where Z is the set of all integers. Define $(a, b)(\alpha, \beta)(c, d)$ by $(a, b)(\alpha, \beta)(c, d) = (a\alpha c, b\beta d)$ for all $(a, b), (c, d) \in S$ and $(\alpha, \beta) \in \Gamma$. Then S is a Γ -semigroup with zero $(0, 0)$. Now for each $(\alpha, \beta) \in \Gamma$, $S_{(\alpha, \beta)}$ is a semigroup where binary operation is defined in $S_{(\alpha, \beta)}$ by $(a, b) \cdot (c, d) = (a, b)(\alpha, \beta)(c, d)$ for all $(a, b), (c, d) \in S$. Now $(2, 3) \in \Gamma$. Consider the semigroup $S_{(2, 3)}$. It is a semigroup with $(0, 0)$. Let $(a, b), (c, d)$ be two non-zero elements of $S_{(2, 3)}$. Then $(a, b)(2, 3)(c, d) = (2ac, 3bd)$ shows that $(a, b) \cdot (c, d) \neq (0, 0)$. We can show that $S_{(2, 3)}$ is a group with zero element. Now, $(0, 2) \in \Gamma$. Let us consider the semigroup $S_{(0, 2)}$. Now $(\frac{1}{2}, 0) \in S$. So, $(\frac{1}{2}, 0)(0, 2)(a, b) = (0, 0)$ shows that $S_{(0, 2)}$ is not a group with zero.

This example shows that the nature of 0 - Γ -group is different from Γ -group. Here " S_α " is a group with " $\cdot$ " may not imply that " S_β " is a group with " $\cdot$ " for all $\beta \in \Gamma$.

Theorem 5.3 [19] A Γ -semigroup S with zero is a 0 - Γ -group if and only if for all $a \in S \setminus \{0\}$ and for all $\alpha \in \Gamma$, $a\alpha S = S\alpha a = S$.

The next theorem shows that the result of Corollary 5.1 was extended to left group.

Theorem 5.4 [19] *Let S be a Γ -semigroup. If $\alpha, \beta \in \Gamma$, then S_α is a left group if and only if S_β is a left group.*

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ON APPLICATION OF SYLOW'S THEOREMS

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Abstract

This paper mainly presents series of research work on Sylow's theorem and examines its formal proof using concrete examples for verification and identifying some of its consequences. Furthermore, some of its applications on a cyclic group order, small (but not simple) groups and infinite groups.

Keywords: Group; Subgroup; Coset; Normal Subgroup; Trivial Subgroup; Quotient group; Simple group; Finite groups; symmetric groups; Alternating groups; Cyclic groups; Sylow's theorem and Sporadic groups.

1 Introduction

The mathematical field which studies the theory of groups in their different forms has evolved in several parallel threads. There are generally three historic roots of the study, namely the theory of algebraic equation, number theory and geometry. Joseph Louis Lagrange, Niels Henrik Abel and Evariste Galois were the early researchers in this field (see [1] and [2]). The initial study of groups could be traced back to the work of Lagrange in the eighteenth century [3]. The work of Augustin Louis Cauchy and Galois are more commonly referred to as the beginning of group theory [1]. The study aroused on the quest to obtain solution of polynomial equations of degree greater than 4. The area is the most broadly-application branch of mathematics that are used to study models, symmetries of certain objects, structures, behaviours of molecules and crystals. Group theory can therefore be viewed as an essential tool in certain areas of chemistry.

In 1770, J. L. Lagrange wrote a paper, which is now known as the fundamental of group theory. In the paper, succinate efforts were made to solve polynomial equations especially that of higher order. Paolo Ruffini (1799) attempted to prove that quantic and higher order functions are impossible to solve and in the process gave the concept of primitive group. Everiste Galois (1830) further studied Ruffini's work and found the groups of permutations of higher order functions. The term group (*groupe* in French) was first used by Galois and discovered the notion of subgroup [4]. Pietro Abbati (1842) discussed the notion of normal subgroups and a solvable primitive group and subsequently identified a subgroup of the affine group of an affine space over a finite field of prime order. In 1844, Augustine-Louis Cauchy investigated group similar to Galois groups and are called today permutation groups (see [5] and [6]). Many number of important theorems in early group theory are also due to Cauchy [5]. The Authur Cayleys work titled "On the theory of groups", depending on the symbolic equation $\theta^n = 1$ gives the first abstract definition of finite groups (see [7] and [8]).

The second root of group theory was number theory. Abelian group structures had been implicitly used in number-theoretical work by Carl Friedrich Gauss (1801), and explicitly by Leopold Kronecker (1882). Gauss made an attempt to refine Fermat's Last Theorem and it was completed by

Ernst Kummer (1843) which led to the introduction of groups describing factorization into prime numbers (see [7] and [8]).

The third root to group theory is geometry. In 19th century, geometry saw a spurt of growth as many new concepts were given and different branches of it came into existence, such as: Algebraic Geometry, Differential Geometry, Projective Geometry etc. Methods of solving geometrical problem, such as synthetic, analytical, metric and projective rose and debated. In the mid-19th century, a problem of linking the geometrical classes to method gave birth to the study of geometrical relations. This area studied the properties of figures invariant under transformation which later became the transformation itself. The connection established between circular transformations and inverse transformation eventually led to Klein's group-theoretic synthesis of geometry. Klein's group theory brought order to geometry and is considered as implicit contribution to group theory. Mathieu (1861, 1873), W. Burnside (1897), Otto Holder (1889), E. H. Moore (1885), Ludwig Sylow (1872) and H. M. Weber (1893) were other group theorists in 19th century (see [5] and [8]).

The concept of group proliferated swiftly during the 1880's and 1890's, although there still appeared a great number papers in areas of permutation and transformation groups (see [2],[3],[4],[5],[7] and [8]). Jodan Holder (1889) was the first to study simple finite group abstractly that were previously considered as permutation or transformation groups. Consequently simple groups with finite number of operation became a matter of concern. By "operation" Holder meant elements and these were used to determine the simple group of order up to 200 which are now called finite simple group (see [8]). Finite group in the 1870 - 1900 period came to popularity as the Sylow's Theorems, Holder's Classification of Groups of Square-free Order, and the early beginnings of character of Theory of Frobenius (see [9] pp. 235).

H. U. Besche and E. Bettina (1999) propounded three new practical algorithms to construct finite groups up to isomorphism. The different methods were developed to solve the isomorphism problem for finite soluble groups and different classes of groups (see [10]). Also B. Eick and M. Horn (2014) described a new approach towards the systematic construction of finite groups up to isomorphism (see [11]). The previous studies by authors

on finite groups and the work of Geord Frobenius (1893) and W. Burnside (1911) are remarkable papers that build up independent theory of finite abelian groups (see [8],[9],[10] and [11]).

The originating studies of abstract group theory concept in abstract setting began to appear with Sylow's theorems, which was proved by Sylow in 1872 for permutation groups in a paper entitled "Neuer Beweis Sylowschen Satzes". Although Frobenius [9] admitted the fact that every finite group can be represented by a group of permutations shows that Sylow's theorem must hold for all finite groups. The Sylow theorems form a fundamental part of finite group theory and have very important applications in the classification of finite simple groups (see [12] and [13]).

In view of the various studies carried out by researchers on group theoretical concepts, Sylow's theorems are a powerful statement about the structure of groups in general and particularly in application of finite group theory because of its effects on the prime decomposition of the cardinality of a finite group G to give statements about the structure of its subgroups, especially, the technique to transport basic number-theoretic information about a group to its group structure. The classification of finite groups becomes a game of finding which combinations/constructions of groups of smaller order that can be applied to construct a group. For example, a typical application of these theorems is in the classification of finite groups of some fixed cardinality, e.g. $|G| = 60$. This observation has motivated us in this paper to revisit some of these applications according to some scholars, and further examples and analysis will be given to make the concepts as clear as possible.

2 Preliminaries

Some fundamental definitions and related theorems of group concept are recalled for this paper.

Definition 2.1 (*Group*) *Let G be a non-empty set and $*$ be a binary operation on G , then the algebraic structure $(G, *)$ is a Group if the following axioms are satisfied:*

a. *Associativity*

For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

b. *Identity element*

There exists an element $e \in G$ such that $e \cdot a = a$ ($a \cdot e = a$) for every $a \in G$.

c. *Inverse element*

For any a in G , there exists an element $b \in G$ such that $a \cdot b = e$ ($b \cdot a = e$), where e is the identity element. For each a , the element b is unique. It is called the inverse of a and is commonly denoted a^{-1} .

Formally, the group is the ordered pair of a set and a binary operation on this set that satisfies the group axioms. The set is called the underlying set of the group, and the operation is called the group operation or the group law.

Definition 2.2 (*Subgroups*) Given a group G under a binary operation $*$, a subset H of G is called a subgroup of G if H also forms a group under the operation $*$. More precisely, H is a subgroup of G if the restriction of $*$ to $H \times H$ is a group operation on H . This is usually denoted $H \leq G$, read as " H is a subgroup of G ".

The trivial subgroup of any group is the subgroup $\{e\}$ consisting of just the identity element.

A proper subgroup of a group G is a subgroup H which is a proper subset of G (that is, $H \neq G$). This is usually represented notationally by $H < G$, read as " H is a proper subgroup of G ". Some authors also exclude the trivial group from being proper (that is, $H \neq \{e\}$).

Definition 2.3 (*Cosets*) Let H be a subgroup of the group G whose operation is written multiplicatively (juxtaposition denotes the group operation). Given an element g of G , the left cosets of H in G are the sets obtained by multiplying each element of H by a fixed element g of G (where g is the left factor). In symbols these are, $gH = \{gh : h \text{ an element of } H\}$ for each g in G . The right cosets are defined similarly, except that the element g is now a right factor, that is, $Hg = \{hg : h \text{ an element of } H\}$ for g in G .

As g varies through the group, it would appear that many cosets (right or left) would be generated.

Definition 2.4 (Normal subgroup) A subgroup N of a group G is called a normal subgroup of G if it is invariant under conjugation; that is, the conjugation of an element of N by an element of G is always in N . The usual notation for this relation is $N \triangleleft G$.

Definition 2.5 (Quotient group) Let N be a normal subgroup of a group G . Define the set G/N to be the set of all left cosets of N in G . That is, $G/N = \{aN : a \in G\}$. Since the identity element $e \in N$, $a \in aN$. Define a binary operation on the set of cosets, G/N , as follows. For each aN and bN in G/N , the product of aN and bN , $(aN)(bN)$, is $(ab)N$. This works only because $(ab)N$ does not depend on the choice of the representatives, a and b , of each left coset, aN and bN .

Definition 2.6 (Simple group) A simple group is a nontrivial group whose only normal subgroups are the trivial group and the group itself. A group that is not simple can be broken S into two smaller groups, namely a nontrivial normal subgroup and the corresponding quotient group. This process can be repeated, and for finite groups one eventually arrives at uniquely determined simple groups.

3 Sylow's Theorem and Applications

Theorem 3.1 (First Sylow's Theorem) [14] A finite group G whose order $|G|$ is divisible by a prime power p^k has a subgroup of order p^k .

Theorem 3.2 (Sylow's second theorem) [15] If H is a p -subgroup of G and P is a Sylow p -subgroup of G , then there exists an element g in G such that $g^{-1}Hg \leq P$. In particular, all Sylow p -subgroups of G are conjugate to each other (and therefore isomorphic), that is, if H and K are Sylow p -subgroups of G , then there exists an element g in G with $g^{-1}Hg = K$.

Theorem 3.3 (Sylow's third theorem) [16] Let q denote the order of any Sylow p -subgroup P of a finite group G . Let n_p denote the number of Sylow p -subgroups of G . Then (a) $n_p = [G : N_G(P)]$ (where $N_G(P)$ is the normalizer of P), (b) n_p divides $|G|/q$, and (c) $n_p \equiv 1 \pmod{p}$.

4 Applications of Sylow's Theorem

Since Sylow's theorem ensures the existence of p -subgroups of a finite group, it is worthwhile to study groups of prime power order more closely. Most of the examples use Sylow's theorem to prove that a group of a particular order is not simple. For groups of small order, the congruence condition of Sylow's theorem is often sufficient to force the existence of a normal subgroup.

Generally speaking we look at groups of order pq , p and q primes with $p < q$. Also Group of order 30, groups of order 20, groups of order p^2q , p and q of distinct primes. If the order $|G| = 60$ and G has more than one Sylow 5-subgroup ($p = 5$), then G is simple. These are some basic examples of decomposition of order of group in the application of Sylow's Theorem.

4.1 Application to Cyclic Group Order

Some non-prime numbers n are such that every group of order n is cyclic. One can show that $n = 15$ is such a number using the Sylow's theorems.

Example 4.1 *Let G be a group of order $15 = 3 \cdot 5$ and n_3 be the number of Sylow 3-subgroups. Then $n_3 | 5$ and $n_3 \equiv 1 \pmod{3}$. The only value satisfying these constraints is 1; therefore, there is only one subgroup of order 3, and it must be normal (since it has no distinct conjugates). Similarly, n_5 must divide 3, and $n_5 \equiv 1 \pmod{5}$; thus it must also have a single normal subgroup of order 5. Since 3 and 5 are coprime, the intersection of these two subgroups is trivial, and so G must be the internal direct product of groups of order 3 and 5, that is, the cyclic group of order 15. Thus, there is only one group of order 15 (up to isomorphism) (see [17] and [18]).*

4.2 Application to small but not simple groups

More complex example involves the order of the smallest simple group that is not cyclic. If G is simple, and $|G| = 30$, then n_3 must divide $10 (= 2 \cdot 5)$, and $n_3 \equiv 1 \pmod{3}$. Therefore, $n_3 = 10$, since neither 4 nor 7 divides 10, and if $n_3 = 1$ then, as above, G would have a normal

subgroup of order 3, and could not be simple. G then has 10 distinct cyclic subgroups of order 3, each of which has 2 elements of order 3 (plus the identity). This means G has at least 20 distinct elements of order 3. As well, $n_5 = 6$, since n_5 must divide $6(= 2 \cdot 3)$, and n_5 must equal $1 \pmod{5}$. So, G also has 24 distinct elements of order 5. But the order of G is only 30, so a simple group of order 30 cannot exist.

Example 4.2 Suppose $|G| = 42 = 2 \cdot 3 \cdot 7$. Here n_7 must divide $6(= 2 \cdot 3)$ and n_7 must equal $1 \pmod{7}$, so $n_7 = 1$. So, as before, G cannot be simple. On the other hand, for $|G| = 60 = 2^2 \cdot 3 \cdot 5$, then $n_3 = 10$ and $n_5 = 6$ is perfectly possible. And in fact, the smallest simple non-cyclic group is A_5 , the alternating group over 5 elements. It has order 60, and has 24 cyclic permutations of order 5, and 20 of order 3. This leads us to the theorems below:

Theorem 4.1 [17] Burnside's $p^a q^b$ theorem states that if the order of a group is the product of one or two prime powers, then it is solvable, and so the group is not simple, or is of prime order and is cyclic. This rules out every group up to order $30(= 2 \cdot 3 \cdot 5)$.

Theorem 4.2 (Wilson's theorem) [17] Wilson's theorem states that a natural number $n > 1$ is a prime number if and only if the product of all the positive integers less than n is one less than a multiple of n . That is (using the notations of modular arithmetic), the factorial $(n-1)! = 1 \times 2 \times 3 \times \dots \times (n-1)$ satisfies $(n-1)! \equiv -1 \pmod{n}$ exactly when n is a prime number. In other words, any number n is a prime number if and only if, $(n-1)! + 1$ is divisible by n .

Proof

It is possible to deduce Wilson's theorem from a particular application of the Sylow's theorems. Let p be a prime. It is immediate to deduce that the symmetric group S_p has exactly $(p-1)!$ elements of order p , namely the p -cycles C_p . On the other hand, each Sylow p -subgroup in S_p is a copy of C_p . Hence it follows that the number of Sylow p -subgroups is $n_p = (p-2)!$. The third Sylow's theorem implies

$$(p-2)! \equiv 1 \pmod{p}. \text{ Multiplying both sides by } (p-1) \text{ gives}$$

$(p-1)! \equiv p-1 \equiv -1 \pmod{p}$ and hence, the result.

4.3 Application to Infinite Groups

There is an analogue of the Sylow's theorems for infinite groups. One defines a Sylow p -subgroup in an infinite group to be a p -subgroup (that is, every element in it has p -power order) that is maximal for inclusion among all p -subgroups in the group. Such subgroups exist by Zorn's lemma. Let $Cl(K)$ denote the set of conjugacy classes of a subgroup $K \subset G$.

Theorem 4.3 [19] *If K is a Sylow p -subgroup of G , and $n_p = |Cl(K)|$ is finite, then every Sylow p -subgroup is conjugate to K , and $n_p \equiv 1 \pmod{p}$.*

The Theorem 4.3.1 elaborated the application to infinite groups in its conjugates and reflexions as in example below.

Example 4.3 *A simple illustration of Sylow subgroups and the Sylow theorems are the dihedral group of the n -gon, D_{2n} . For n odd, $2 = 2^1$ is the highest power of 2 dividing the order, and thus subgroups of order 2 are Sylow subgroups. These are the groups generated by a reflection, of which there are n , and they are all conjugate under rotations; geometrically the axes of symmetry pass through a vertex and a side and are shown in the figures below.*

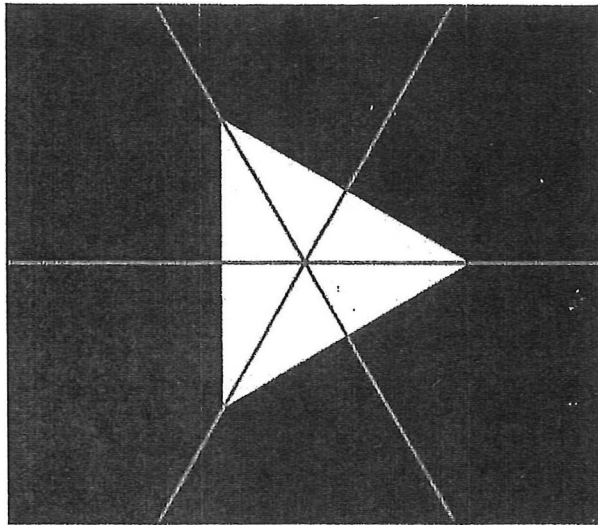


Figure 1: In D_6 all reflection are conjugate, as reflection corresponds to Sylow 2-subgroup

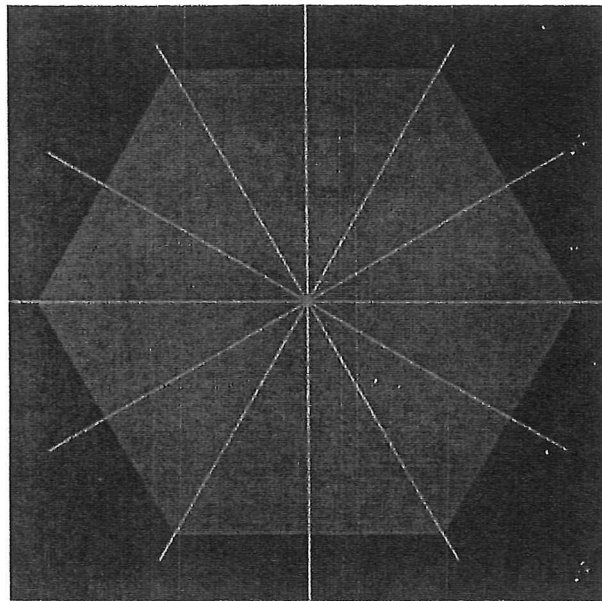


Figure 2: In D_{12} reflection no longer corresponds to sylow 2- subgroups, and fall into two conjugacy class

4.4 Explanation

If n is even, then 4 divides the order of the group, and the subgroups of order 2 are no longer Sylow subgroups, and in fact they fall into two conjugacy classes, geometrically according to whether they pass through two vertices or two faces. These are related by an outer automorphism, which can be represented by rotation through π/n , half the minimal rotation in the dihedral group.

Another example are the Sylow p -subgroups of $GL_2(F_q)$, where p and q are primes ≥ 3 and $p \equiv 1 \pmod{q}$ which are all abelian. The order of $GL_2(F_q)$ is $(q^2 - 1)(q^2 - q) = (q)(q + 1)(q - 1)2$. Since $q = p^n m + 1$, the order of $GL_2(F_q) = p^{2n} m^1$. Thus, the order of the Sylow p -subgroups is p^{2n} .

5 Conclusion

The Sylows theorem has a number of applications to many areas of abstract algebra and the paper has been able to present these applications of the theorems with very clear examples using figures and actual numbers to intuitively analyze the theorems.

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THE PRIME PRINCIPLE IN CLUSTERS AND NANOSTRUCTURES

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Abstract

Why we have five fingers. We suggest two principles: (1) the prime principle and (2) the symmetric principle. We prove that 1, 3, 5, 7, 11, 23, 47, and 2, 4, 6, 10, 14, 22, 46, 94 are the most stable numbers, which are the basic building-blocks in clusters and nanostructures. The prime principle is the mathematical foundations for clusters and nanosciences. It is a theory of everything.

Why do we have five fingers? We suggest two principles [1-8]:

(1) **The prime principle.** A prime number is irreducible over the integer field, it seems therefore natural to associate it with the most stable cluster in nature.

(2) **The symmetric principle.** Cluster of two stable prime numbers is then stable symmetric system in nature.

According to Euler function and stable group theory [3, 5], we have

$$2P_1 + 1 = P_2. \quad (1)$$

If P_1 is the most stable prime, then P_2 also is the most stable prime. For example,

$2 \times 1 + 3 = \text{prime}$, $2 \times 3 + 1 = 7 = \text{prime}$, $2 \times 7 + 1 = 3 \times 5 = \text{no prime}$;

$2 \times 2 + 1 = 5 = \text{prime}$, $2 \times 5 + 1 = 11 = \text{prime}$, $2 \times 11 + 1 = 23 = \text{prime}$,

$2 \times 23 + 1 = 47 = \text{prime}$, $2 \times 47 + 1 = 5 \times 19 = \text{no prime}$. From above calculations we come to conclusion that 1, 3, 5, 7, 11, 23, 47, and 2, 4, 6, 10, 14, 22, 46, 94 are the most stable numbers, which are the basic building-blocks in clusters and nanostructures. Trigonal, tetragonal, pentagonal, hexagonal, heptagonal, decagonal, hendecagonal, 14-gonal, 22-gonal, 23-gonal, 46-gonal, 47-gonal and 94-gonal are the most stable clusters, which are the basic building-blocks of polyhedra. Therefore tetrahedra, hexahedra, octahedra, dodecahedra and icosahedra are the most stable clusters [9]. The prime principle is the mathematical foundations for clusters and nanosciences.

We can cite many examples in illustration of this theory below.

Example 1. C_{60} : Buckminsterfullerene. It is icosahedron, a polygon with 60 vertices and 32 faces, 12 of which are pentagonal and 20 hexagonal [10], which is the most stable cluster.

The year 1985 is often given as the beginning of nanosciences. Indeed, the technical improvements follow the discovery of other new members of the fullerene family, such as C_{76} , C_{78} , C_{82} , C_{84} , and their isomers.

Example 2. In the periodic table of the elements, electric subshells: 2, 6, 10, and 14 are the most stable subshells but 18 is unstable subshell. We prove that rare

gases (He ($Z=2$), Ne($Z=10$), Ar($Z=18$), Kr($Z=36$), Xe($Z=54$), and Rn($Z=86$)) are the most stable clusters, the heaviest element that occurs naturally is uranium with an atomic number of 92 and the island of stability does not exist [1, 4, 7, 8].

Example 3. In the human chromosomes most people thought one of the these two numbers ($2n=47$ and $2n=48$) must be right. Since 1882 when Plemming[11] observed cell divisions in human cornea, it took 74 years to determine the number of human chromosomes. The success finally achieved in 1956 [12], Tjio and Leven started the world with new finding of $2n=46$, which is now recognized as the correct number, which is the most stable number. The year 1956 is often given as the beginning of modern human cytogenetics. Human is so advanced because we have 46 chromosomes (23 pairs) in a cell [5].

Indeed, biologist in general have paid too little attention to the quantitative aspects of their work in the past few hectic decades. That is understandable when there are so many interesting (and important) data to be gathered. But we are already at the point where deeper understanding of how, say, cell function is impeded by the simplification of reality now commonplace in cell biology and genetics-and by the torrent of data accumulating everywhere. Simplification? In genetics, it is customary to look for (and to speak of) the “function” of a newly discovered gene. But what if most of the genes in the human genome, or at least their protein products, have more than one function, perhaps even mutually antagonistic ones? Plainlanguage accounts of cellular events are then likely to be misleading or meaningless. Using the prime principle one can carry out the quantitative analyses in biological experiments. But it does not take 74 years like human chromosomes.

Example 4. A new evolution theory in the biology. The evolution of the living organisms starts with mutant of the prime number. The living organism mutates from a prime number system to another new one which may be produced a new species to raise up seed and its structure tended to stability in given environment [5].

Example 5. The most important advance in biology in decade has been the discovery of the RNAi (short for RNA interference). We postulate that the RNAi is 3-bp, 5-bp, 7-bp, 11-bp 23-pb and 47-bp in length, which are the most stable clusters.

Example 6. DNA commonly contains the purines adenine (A) and guanine (G),

the pyrimidines cytosine (C) and thymine (T), the sugar deoxyribose, and phosphate. RNA contains the purine adenine and guanine, and the pyrimidines cytosine and uracil (U), together with the sugar ribose, and phosphate. Because the four is the most stable number.

Example 7. The major light-harvesting complex of photosystem II (LHC-II) serves as the principal solar energy collector in the photosynthesis of green plants and presumably also functions in photoprotection under high-light conditions. Spinach structure determination of LHC-II is organized in an icosahedral cluster [13], which is the most stable.

Example 8. Diakonov *et al* predicted [14] and Naknao *et al* [15] searched a penta-quark particle. We postulate that the stable particle is made of n -quarks, where n is the most stable number.

Example 9. The EPR paper [16] introduced quantum entanglement and laid the groundwork not only for John Bell's investigation into nonlocality, but also for the contemporary development of quantum information theory (computing, cryptography and teleportation). The five-photon entanglement has been achieved experimentally [17]. We postulate experimental demonstration of n -photon entanglements, where n is the most stable number.

Example 10. In viruses there are many icosahedral clusters. Using this theory one may study the structures of viruses and make vaccines.

Example 11. Musical notes are 7 Arabic numerals: 1, 2, 3, 4, 5, 6, and 7. Every is the most stable number.

Example 12. Light is compound of the seven colours: red, orange, yellow, green, blue, indigo and violet. Every colour is the most stable cluster.

Example 13. Kepler conjectured four questions concerning nature's forms: why honeycombs are formed as hexagons, why the seeds of pomegranates are shaped as dodecahedra, why the petals of flowers are most often grouped in five, why snow crystal are six-cornered [18]. Because which are the most stable clusters.

Example 14. Fibonacci's numbers $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ and Gold

ratio $\alpha = \frac{\sqrt{5}+1}{2} = 1.618$ are both beautiful formulas in nature. Why this should

be so is a mystery even today [18]. Because the five is the most stable number.

Example 15. In Chinese poem there are five and seven characters. In English poem there is the iambic pentameter, Although the languages are different, the human brains are the same. Using the five and seven the brain structures and the nervous systems can be studied. For example, there are Tyr-Gly-Gly-Phe-Met, Met-enkephalin, and Tyr-Gly-Gly-Phe-Leu, Leu-enkephalin in brain.

Example 16. Pentacene ($C_{14}H_{22}$) consists of five fused aromatic-benzenelike-rings [19]. It is one of the most promising candidates for organic electronics, in part because of its chemical and thermal stability, and in part because its planar shape facilitates crystalline packing. We postulate that there is n -cenes, where n is the most stable number, which consists of n -fused aromatic-benzenelike-rings.

Example 17. In Al-Mn alloys there are the icosahedral phase [20], which is the most stable cluster.

Example 18. We postulate the existences of single stranded DNAs and RNAs, double-stranded DNAs and RNAs, tri-stranded DNAs and RNAs, tetra-stranded DNAs and RNAs, penta-stranded DNAs and RNAs, hexa-stranded DNAs and RNAs, as well as hepta-stranded DNAs and RNAs in biology, which are the most stable clusters.

Example 19. Extensive computer experiments on a metal encapsulated silicon and germanium clusters led to the finding of novel fullerenelike, cubic, Frank-Kasper polyhedral, icosahedral as well as other forms of clusters [21], which are the most stable.

In number theory we have the Cunningham chain of the first kind [8]

$$2P_i + 1 = P_{i+1}, \quad (2)$$

where $i = 1, 2, 3, \dots, P_1 > 47$.

If P_i is the most stable prime, then P_{i+1} also is the most stable prime. For (2) one can find the big primes, which form the most stable clusters. It is the biological and chemical mathematics.

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DUAL B-ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract

The concept of dual B-Almost distributive fuzzy lattice ($dualB-ADFL$) in terms of its principal ideal fuzzy lattice is introduced. We also prove the equivalency of dual B-ADL and dual B-ADFL and The relations of B-ADFL and Dual B-ADFL are investigated.

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1 Introduction

The concept of an Almost distributive lattice was introduced by Swamy and Rao in [8]. It is an algebraic structure which satisfies almost all the properties of a distributive lattice with the smallest element except the commutativity of the operations $\wedge$ and $\vee$ and the right distributive of $\vee$ over $\wedge$.

The concept of pseudo-complementation in an Almost distributive lattice was introduced by Swamy, Rao and Nanaji Rao in [11] and the notion of Birkhoff center of an Almost distributive lattice with maximal element was introduced by Swamy and Ramesh in [9].

A Heyting algebra is a distributive lattice in which, for any $a, b \in R$, the largest element $a \rightarrow b$ in R exists with the property $a \wedge (a \rightarrow b) \leq b$. Heyting algebra serve as the algebraic models of propositional intuitionistic logic in the same way Boolean algebras model propositional classical logic. The concept of Heyting Almost distributive lattice (HADL) was introduced by Rao, Berhanu and Ratnamani in [6].

In studying the properties of a post algebra, G. Epstein and A. Horn in [3] introduced the concept of B-algebra as a bounded distributive lattice with center $B(R)$ in which for any $a, b \in R$, the largest element $a \Rightarrow b$ in $B(R)$ exists with the property $a \wedge (a \Rightarrow b) \leq b$. The concept of fuzzy set was first introduced by Zadeh in [13], and Sanchez in [12] use to define and study fuzzy relations. As a continuation of these studies, we use fuzzy relation, fuzzy poset and fuzzy lattice defined in [4].

In this paper, we introduce the concept of a dual B-Almost distributive fuzzy lattice (dual B-ADFL) as an extension of dual B-ADL in which the set of all principal ideal fuzzy lattice of (R, A) is a B-fuzzy algebra. Here (R, A) represents an ADFL which again become a dual B-ADFL. $x \in (R, A) \Leftrightarrow x \in R$.

$B_A(R)$ represent a Birkhoff center of (R, A) and $A : R \times R \rightarrow [0, 1]$.

2 preliminaries

Definition 2.1 [7] *An algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is said to be an Almost distributive lattice (ADL) with 0 if, it satisfy the following condition:*

1. $a \vee 0 = a$.
2. $0 \wedge a = 0$.
3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
6. $(a \vee b) \wedge b = b$ for all $a, b, c \in R$.

Lemma 2.2 [7] *For any $a, b \in R$ we have*

1. $a \wedge 0 = 0$ and $0 \vee a = a$
2. $a \wedge a = a, a \vee a = a$

3. $(a \wedge b) \vee b = b, a \vee (b \wedge a) = a$ and $a \wedge (a \vee b) = a$
4. $a \wedge b = b \Leftrightarrow a \vee b = a$
5. $a \wedge b = a \Leftrightarrow a \vee b = b$
6. $a \wedge b \leq b$ and $a \leq a \vee b$
7. $a \wedge b = b \wedge a$, whenever $a \leq b$
8. $a \vee (b \vee a) = a \vee b$

Theorem 2.3 [7] For any $a, b \in R$ we have:

1. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
2. $\wedge$ is associative in R
3. $a \wedge b \wedge c = b \wedge a \wedge c$

From the above it follows that for any $a \in R$ the set $\{a \wedge x | x \in R\}$ forms a bounded distributive lattice.

In particular, we have $((a \wedge b) \vee c) \wedge x = ((a \vee c) \wedge (b \vee c)) \wedge x$ for all $a, b, c, x \in R$.
An element $m \in R$ is said to be maximal if $m \leq x$ implies $m = x$.

Lemma 2.4 [7] Let R be an ADL with 0 and $m \in R$. Then the following are equivalent:

1. m is a maximal element with respect to a poset $(R, \leq)$.
2. $m \vee x = m$ for all $x \in R$.
3. $m \wedge x = x$ for all $x \in R$.

Theorem 2.5 [7] The following are equivalent for any ADL with 0:

1. R is relatively complemented.
2. Given $x, y \in R$, there exist $a \in R$ such that $a \wedge x = 0$ and $a \vee x = x \vee y$
3. For any $x \in R$, the interval $[0, x]$ is complemented.

Lemma 2.6 [7] a relatively complemented ADL R is associative.

Lemma 2.7 [7] For any $a, b \in R$, the following hold:

1. $[a] \cap [b] = (a \wedge b) = (b \wedge a)$.
2. $[a] \vee [b] = (a \vee b) = (b \vee a)$.

Theorem 2.8 [7] For any $a, b \in R$ we have the following:

1. $[a] = \{a \wedge x \mid x \in R\}.$
2. $a \in [b] \Leftrightarrow b \wedge a = a \Leftrightarrow [a] \subseteq [b].$

Definition 2.9 [8] Let $(R, \vee, \wedge, 0, m)$ be an ADL with maximal element m . Suppose $\rightarrow$ is a binary operation on R satisfying the following condition:

1. $a \rightarrow a = m.$
2. $(a \rightarrow b) \wedge b = b.$
3. $a \wedge (a \rightarrow b) = a \wedge b \wedge m.$
4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c).$
5. $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c),$ for all $a, b, c \in R$. Then $(R, \vee, \wedge, \rightarrow, 0, m)$ is an HADL

Lemma 2.10 [5] Let $(R, \vee, \wedge, \rightarrow, 0, m)$ be an HADL, $x, y, a \in R$ and $x \leq y$. Then the following condition hold:

1. $a \rightarrow x \leq a \rightarrow y.$
2. $y \rightarrow a \leq x \rightarrow a.$

Definition 2.11 [4] Let X be a non-empty set. A function $A: X \times X \rightarrow [0, 1]$ is said to be fuzzy partial order relation if it satisfies the following:

1. $A(x, x) = 1.$ That is A is reflexive.
2. $A(x, y) > 0$ and $A(y, x) > 0$ implies that $x = y$. That is A is antisymmetric.
3. $A(x, z) \geq \sup_{y \in X} \min[A(x, y), A(y, z)] > 0.$ That is A is transitive.

If A is a fuzzy partial order relation in a set X , then (X, A) is called a fuzzy partial order relation or fuzzy poset.

Definition 2.12 [4] Let (X, A) be a fuzzy poset. Then (X, A) is a fuzzy lattice if and only if $x \vee y$ and $x \wedge y$ exists for all $x, y \in X$.

Definition 2.13 [4] Let (X, A) be a fuzzy lattice. Then (X, A) is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z),$ for all $x, y, z \in X$.

Definition 2.14 [4] Let (X, A) be a fuzzy lattice and $Y \subseteq X$. Then Y is an ideal of (X, A) .

1. If $x \in X$ and $y \in Y$ and $A(x, y) > 0,$ then $x \in Y$.
2. If $x, y \in Y,$ then $x \vee y \in Y$.

Definition 2.15 [8] Let R be an ADL with maximal element m and $B(R) = \{a \in R \mid a \wedge b = 0 \text{ and } a \vee b \text{ is maximal, for some } b \in R\}.$

Then $(B(R), \vee, \wedge)$ is a relatively complemented ADL and it is called Birkhoff center of R .

Definition 2.16 [2] Let R be a distributive lattice with $0, 1$ and $B(R)$ be the Birkhoff center of R . If for any $a, b \in R$ there exist a greatest element $y \in B(R)$ such that $a \wedge y \leq b$, then R is called a B -algebra. This element y is denoted by $a \Rightarrow b$.

Definition 2.17 [6] An ADL R is called a B -ADL if its principal ideal lattice $PI(R)$ is a B -algebra.

Theorem 2.18 [6] Let R be an ADL with Birkhoff center $B(R)$. Then R is a B -ADL if and only if for any $a, b \in R$, there exists $y \in B(R)$ satisfying the following condition:

$$R_1: b \wedge a \wedge y = a \wedge y.$$

$$R_2: \text{If } z \in B(R) \text{ such that } b \wedge a \wedge z = a \wedge z, \text{ then } y \wedge z = z.$$

Definition 2.19 Let R be an ADL with Birkhoff center $B(R)$. Then R is called a dual B -ADL, if for any $a, b \in R$, there exist $y \in B(R)$ satisfy the following:

$$1. (a \vee y) \wedge b = b$$

$$2. \text{If } z \in B(R) \text{ such that } (a \vee z) \wedge b = b, \text{ then } z \wedge y = y.$$

Definition 2.20 [2] Let R be a distributive lattice with $0, 1$ and $B(R)$ be the Birkhoff center of R . If for any $a, b \in R$ there exist a least element $y \in B(R)$ such that $b \leq a \vee y$, then R is called a dual B -algebra. This element y is denoted by $a \Leftarrow b$.

Lemma 2.21 Let R be an ADL with a maximal element m . Then $z \wedge m = m \wedge z$, for any $z \in R$.

Definition 2.22 [1] Let $(R, \vee, \wedge, 0)$ be an algebra of type $(2, 2, 0)$ and we call (R, A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following condition satisfied:

$$1. A(a, a \vee 0) = A(a \vee 0, a) = 1.$$

$$2. A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1.$$

$$3. A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1.$$

$$4. A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1.$$

$$5. A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1.$$

$$6. A((a \vee b) \wedge b, b) \approx A(b, (a \vee b) \wedge b) = 1, \text{ for all } a, b, c \in R.$$

Definition 2.23 [1] Let (R, A) be an ADFL. Then for any $a, b \in R$, $a \leq b$ if and only if $A(a, b) > 0$.

Definition 2.24 Let (R, A) be an ADFL with a maximal element m and $B_A(R) = \{a \in R \mid A(a \wedge b, 0) > 0 \text{ and } A((a \vee b) \vee x, a \vee b) > 0 \text{ for some } b \in R \text{ and for all } x \in R\}$. Then $(B_A(R), \vee, \wedge)$ is a relatively complemented ADFL and $B_A(R)$ is called the Birkhoff center of an ADFL (R, A) .

3 Main Results

Definition 3.1 An ADFL (R, A) is called a B-ADFL if its principal ideal fuzzy lattice $(PI(R), A)$ is a B-fuzzy algebra.

Definition 3.2 Let (R, A) be an ADFL with Birkhoff center $B_A(R)$. Then (R, A) is called dual B-ADFL, if for any $a, b \in R$ there exists $y \in B_A(R)$ satisfying the following:

d_1 : $A(b, (a \vee y) \wedge b) > 0$.

d_2 : If $z \in B_A(R)$ such that $A(b, (a \vee z) \wedge b) > 0$, then $A(y, z \wedge y) > 0$.

Theorem 3.3 R is a dual B-ADL with Birkhoff center $B(R)$ if and only if (R, A) is a dual B-ADFL with Birkhoff center $B_A(R)$.

Theorem 3.4 Let (R, A) be a B-ADFL with Birkhoff center $B_A(R)$. Then (R, A) is a dual B-ADFL in which for any $a, b \in R$, $b \Leftarrow a$ is the complement of $a \Rightarrow b$ in $[0, m]$.

Proof. Since $A(a \wedge (a \Rightarrow b), b \wedge m) > 0$, we get $A(a \wedge m, [(a \Rightarrow b)' \vee b] \wedge m) > 0$, where $(a \Rightarrow b)'$ is the complement of $a \Rightarrow b$ in $[0, m]$.

Hence $A(a \wedge m, [a \wedge ((a \Rightarrow b)' \vee b)] \wedge m) = A([a \wedge ((a \Rightarrow b)' \vee b)] \wedge m, a \wedge m) = 1$. Thus $A(a, ((a \Rightarrow b)' \vee b) \wedge a) = A(((a \Rightarrow b)' \vee b) \wedge a, a) = 1$,

since $(a \Rightarrow b) \wedge b = b \Rightarrow ((a \Rightarrow b)' \vee b) \wedge a = a$.

Suppose $y \in B_A(R)$ such that $A(a, y \vee b) > 0$, then $A(a \wedge m, (y \wedge m) \vee (b \wedge m)) > 0$. So that $A((a \wedge m) \wedge (y \wedge m)', b \wedge m) > 0$ and hence $A((y \wedge m)', a \Rightarrow b) > 0$. Thus $A((a \Rightarrow b)', y \wedge m) > 0$. Hence (R, A) is a dual B-ADFL in which $b \Leftarrow a$ is $(a \Rightarrow b)'$.

Definition 3.5 Let (R, A) be a distributive fuzzy lattice with $0, 1$ and Birkhoff center $B_A(R)$. For each $a, b \in (R, A)$, there exist a greatest element $y \in B_A(R)$ such that $A(b, a \vee y) > 0$, then (R, A) is called a dual B-fuzzy algebra. This element y if exist is unique and denoted by $a \Leftarrow b$.

Theorem 3.6 Let (R, A) be an ADFL with a maximal element m and Birkhoff center $B_A(R)$. Then (R, A) is a B-ADFL if and only if (R, A) is a dual B-ADFL.

Proof. Suppose (R, A) be an ADFL with maximal element and Birkhoff center $B_A(R)$. Assume (R, A) is a B-ADFL. Since $A(a \wedge (a \Rightarrow b), b \wedge m) > 0$, we get $A(a \wedge m, [(a \Rightarrow b)' \vee b] \wedge m) > 0$, where $(a \Rightarrow b)'$ is the complement of $a \Rightarrow b$ in $[0, m]$.

Thus $A(a \wedge m, [a \wedge ((a \Rightarrow b)' \vee b)] \wedge m) = A([a \wedge ((a \Rightarrow b)' \vee b)] \wedge m, a \wedge m) = 1$ and hence $A(a, ((a \Rightarrow b)' \vee b) \wedge a) = A((a \Rightarrow b)' \vee b) \wedge a, a) = 1$.

Suppose $y \in B_A(R)$ such that $A(a, y \vee b) > 0$. Then $A(a \wedge m, (y \vee b) \wedge m) = A(a \wedge m, (y \wedge m) \vee (b \wedge m)) > 0$.

Now, $A((a \wedge m) \wedge (y \wedge m)', (y \wedge m)' \wedge (y \wedge m) \vee (b \wedge m))$

$= A((a \wedge m) \wedge (y \wedge m)', 0 \vee (b \wedge m))$

$= A((a \wedge m) \wedge (y \wedge m)', b \wedge m) > 0$ and hence $A((y \wedge m)', a \Rightarrow b) > 0$.

Thus $A((a \Rightarrow b)', y \wedge m) > 0$. Therefore (R, A) is a dual B-ADFL in which $b \Leftarrow a$ is $(a \Rightarrow b)'$.

Conversely, let (R, A) be a dual B-ADFL. We need to show (R, A) is a B-ADFL. Let (R, A) be a dual B-ADFL. Then the principal ideal fuzzy lattice of $(PI(R), A)$ is a dual B-fuzzy algebra. First we have to show $(PI(R), A)$ is a B-fuzzy algebra.

(1) Let $y \in B_A(R)$ such that $(a)_A \Rightarrow (b)_A = (y)_A$. We have $(a)_A \cap (y)_A = (a \wedge y)_A \subseteq (b)_A$

$$\Rightarrow a \wedge y \in (b)_A$$

$$\Rightarrow A(a \wedge y, b \wedge a \wedge y) > 0$$

Since $b \wedge a \wedge y \leq a \wedge y$, we get $A(b \wedge a \wedge y, a \wedge y) > 0$.

Hence $b \wedge a \wedge y = a \wedge y$ by antisymmetry property of A .

Therefore $A(a \wedge y, b \wedge a \wedge y) > 0$.

(2) Let $z \in B_A(R)$ such that $A(a \wedge z, b) > 0$. Then $(a \wedge z)_A \subseteq (b)_A$ and hence $A(z, y \wedge z) > 0$.

Thus the two condition hold. Let $(a), (b) \in (PI(R), A)$ and $a, b \in R$. Then there exist $y \in B_A(R)$ satisfy the above conditions. We get $(y) \in B_A(PI(R))$. By condition (1), we have $A(a \wedge y, b) > 0$ and hence $(a \wedge y)_A \subseteq (b)_A$.

Suppose $(z) \in B_A(PI(R))$ such that $(a \wedge z)_A \subseteq (b)_A$. Then $z \in B_A(R)$. and $a \wedge z \in (b)_A$

$$\Rightarrow A(a \wedge z, b \wedge a \wedge z) > 0.$$

Since $b \wedge a \wedge z \leq a \wedge z \Rightarrow A(b \wedge a \wedge z, a \wedge z) > 0$ and hence $A(z, y \wedge z) > 0$.

$$\Rightarrow z \in (y)_A \text{ which implies that } (z)_A \subseteq (y)_A.$$

Thus $(PI(R), A)$ is a B-fuzzy algebra. Consequently, (R, A) is a B-ADFL.

Theorem 3.7 A dual B-ADL R with $0, 1$ and Birkhoff center $B(R)$ is a dual B-algebra if and only if (R, A) with $0, 1$ and Birkhoff center $B_A(R)$ is a dual B-fuzzy algebra.

Definition 3.8 An ADFL (R, A) with 0 and maximal element m is called a dual B-ADFL if its principal ideal fuzzy lattice $(PI(R), A)$ is a dual B-fuzzy algebra.

Theorem 3.9 Let (R, A) be an ADFL with zero and maximal element m and Birkhoff center $B_A(R)$. Then (R, A) is dual B-ADFL if and only if for any $a, b \in R$, there exists an element $y \in B_A(R)$ satisfying the following conditions:

$$d_1: A(b, a \vee y) > 0.$$

$$d_2: \text{If } z \in B_A(R) \text{ such that } A(b, a \vee z) > 0, \text{ then } A(y, z \wedge y) > 0.$$

Proof. Suppose (R, A) is a dual B-ADFL and $a, b \in R$. Then $(PI(R), A)$ is a dual B-fuzzy algebra.

(1) There exist $y \in B_A(R)$ such that $(a)_A \Rightarrow (b)_A = (y)_A$ we have

$$(b)_A \subseteq (a)_A \vee (y)_A \Leftrightarrow ((a)_A \vee (y)_A) \wedge (b)_A = (b)_A \quad (1)$$

$$\Leftrightarrow (a \vee y)_A \wedge (b)_A = (b)_A \Rightarrow b \in (a \vee y)_A \Rightarrow b \leq a \vee y. \quad (2)$$

Hence $A(b, a \vee y) > 0$.

(2) Let $z \in B_A(R)$ such that $A(b, a \vee z) > 0$. Then $(b)_A \subseteq (a)_A \vee (z)_A$ and hence $(y)_A \subseteq (z)_A$ which implies $y \in (z)_A \Rightarrow A(y, z \wedge y) > 0$.

Since $z \wedge y \leq y$ we have $A(z \wedge y, y) > 0$. So that we get $z \wedge y = y$.

Therefore $A(y, z \wedge y) > 0$.

Conversely, Suppose (R, A) satisfies d_1 and d_2 . Let $(a), (b) \in (PI(R), A)$, where $a, b \in R$. Then there exists $y \in B_A(R)$ satisfying d_1 and d_2 we get $(y) \in B_A(PI(R))$. By d_1 , we have $A(b, a \vee y) > 0$ and hence $(b)_A \subseteq (a)_A \vee (y)_A$. Suppose $z \in B_A(R)$ such that $(b)_A \subseteq (a)_A \vee (z)_A$. Then $z \in B_A(R)$ and since $b \in (a)_A \vee (z)_A = (a \vee z)_A \Rightarrow b \leq a \vee z$. So that we have $A(b, a \vee z) > 0$ and hence $A(y, z \wedge y) > 0$ which implies that $(y)_A \subseteq (z)_A$. As a result we have $(PI(R), A)$ is a dual B-fuzzy algebra.

Thus (R, A) is a dual B-ADFL.

Theorem 3.10 Let (R, A) be an ADFL with a maximal element m and Birkhoff center $B_A(R)$. Then the following are equivalent :

1. (R, A) is a B-ADFL.
2. (R, A) is a dual B-ADFL.
3. $([0, a], A)$ is a B-fuzzy algebra for all $a \in B_A(R)$.
4. $[0, m]$ is a B-algebra.

Proof. (1) $1 \Leftrightarrow 2$ is given by theorem 3.6.

To prove $1 \Rightarrow 3$. Suppose (R, A) is a B-ADFL and $b \in B_A([0, a])$. Then for $c, b \in [0, a]$, define $A(b \Rightarrow_a c, (b \Rightarrow c) \wedge a) = A((b \Rightarrow c) \wedge a, (b \Rightarrow c) \wedge a) = 1$. Then $b \Rightarrow_a c \in B_A([0, a])$. Since $(b \Rightarrow c) \wedge a \leq a$. If $d \in B_A([0, a])$, then we get $A(d, e \wedge a) = A(e \wedge a, d) = 1$ for some $e \in B_A([0, a])$.

Now, $A(b \wedge d, c) > 0$ if and only if $A(b \wedge e \wedge a, c) > 0$ if and only if $A(e \wedge a, (b \Rightarrow c) \wedge a) > 0$ if and only if $A(d, b \Rightarrow_a c) > 0$.

Hence $([0, a], A)$ is a B-fuzzy algebra.

To prove $3 \Rightarrow 4$ hold. Assume $([0, a], A)$ is a B-fuzzy algebra. For any $c, b \in [0, a]$, there exist $d \in [0, a]$ such that $A(b \wedge d, c) > 0 \Rightarrow b \wedge d \leq c$

Imply that $c \wedge (b \wedge d) = b \wedge d$. Hence $[0, a]$ is a B-algebra. Take $a \leq m$. So that $[0, a] \subseteq [0, m]$ and define $A(c \Rightarrow_m b, (c \Rightarrow b) \wedge m) = A((c \Rightarrow b) \wedge m, c \Rightarrow_m b) = 1$, for any $c, b \in [0, m]$. Let $c, b, d \in [0, m]$ and $c \wedge d \leq b$. Then

$$A(c \Rightarrow (c \wedge d), c \Rightarrow b) > 0 \quad (3)$$

$$A((c \Rightarrow c) \wedge (c \Rightarrow d), c \Rightarrow b) > 0 \quad (4)$$

$$A(m \wedge (c \Rightarrow d), c \Rightarrow b) > 0 \Rightarrow A(c \Rightarrow d, c \Rightarrow b) > 0 \quad (5)$$

$$A((c \Rightarrow d) \wedge m, (c \Rightarrow b) \wedge m) > 0 \quad (6)$$

$$\Rightarrow A((c \Rightarrow d) \wedge m, c \Rightarrow_m b) > 0 \quad (7)$$

$$\Rightarrow A(d \wedge (c \Rightarrow d) \wedge m, d \wedge (c \Rightarrow_m b)) > 0 \quad (8)$$

$$\Rightarrow A((c \Rightarrow d) \wedge d \wedge m, d \wedge (c \Rightarrow_m b)) > 0 \text{ and } \quad (9)$$

$$A(d \wedge (c \Rightarrow_m b), c \Rightarrow_m b) > 0 \quad (10)$$

$$A(d \wedge m, c \Rightarrow_m b) > 0 \quad (11)$$

Thus $d \leq c \Rightarrow_m b$.

Conversely, assume that $d \in [0, m]$ and $A(d, c \Rightarrow_m b) > 0$. Then

$$A(c \wedge d, c \wedge (c \Rightarrow_m b)) \quad (12)$$

$$= A(c \wedge d, c \wedge (c \Rightarrow b) \wedge m) \quad (13)$$

$$A(c \wedge d, c \wedge b \wedge m \wedge m) = A(c \wedge d, c \wedge b \wedge m) \quad (14)$$

$$A(c \wedge d, b \wedge m) = A(c \wedge d, m \wedge b) = A(c \wedge d, b) > 0 \quad (15)$$

since $d \leq c \Rightarrow_m b \Rightarrow b \wedge m \Rightarrow c \wedge d \leq b \wedge m$.

Hence $c \wedge d \leq b$. So that we have $[0, m]$ is a B-algebra.

To prove $4 \Rightarrow 1$. Suppose $([0, m], \vee, \wedge, \Rightarrow_m, 0, m)$ is a B-fuzzy algebra. For any $a, b \in (R, A)$, define $A(a \Rightarrow b, (a \wedge m) \Rightarrow_m (b \wedge m)) = A((a \wedge m) \Rightarrow_m (b \wedge m), a \Rightarrow b) = 1$. Then $A((a \wedge m) \wedge (a \wedge m) \Rightarrow_m (b \wedge m), b \wedge m) > 0$. since $A(a \wedge (a \Rightarrow b), b \wedge m) > 0$. If $c \in B((R, A))$ and $A(a \wedge c \wedge m, b \wedge m) > 0$, then $A(c \wedge m, (a \wedge m) \Rightarrow_m (b \wedge m) = a \Rightarrow b) > 0$

$\Leftrightarrow A(c \wedge m, a \Rightarrow b) > 0$. Therefore (R, A) is a B-ADFL.

4 Conclusion

In this paper we introduce dual B-ADFL as extension of dual B-ADL and different characterization theorems have been proved. We also proved B -ADFL if and only if dual B -ADFL, Basic definitions of dual B -ADFL and dual B -fuzzy algebra are introduced.

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Data Availability

The data used to support the findings of this study are included by citation within the study of the article. No figures, photo and pictures in this research paper and there are tables.

Conflict of interest

The authors declared that there is no conflict of interest on this research paper.

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THE FULL BOCHNER THEOREM ON REAL REDUCTIVE GROUPS

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Abstract

The major results leading to the spherical Bochner theorem and its (spherical) extension, were made possible through the spherical transform theory of Trombi-Varadarajan and were greatly controlled by the non-availability of the full (non-spherical) Harish-Chandra Fourier transform theory on a general connected semisimple Lie group, G . Sequel to a recently announced results where the full image of the Schwartz-type algebras, $C^p(G)$, under the full scalar-valued Harish-Chandra Fourier transform is explicitly computed to be $C^p(\widehat{G}) := \{(\widehat{\xi}_1)^{-1} \cdot h \cdot (\widehat{\xi}_1)^{-1} : h \in \widetilde{\mathcal{Z}}(\mathfrak{F}^\epsilon)\}$ with $\widetilde{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ given as the Trombi-Varadarajan image of $C^p(G//K)$ ($0 < p \leq 2$), the present paper now gives the full Bochner theorem for G by lifting the spherical results to full non-spherical status. An extension of the full Bochner theorem to all of $C^p(G)$, $1 \leq p \leq 2$, is then established. It is also conjectured that every positive-definite distribution T on G which corresponds to a Bochner measure μ on $\mathfrak{F}^\epsilon$ extends uniquely to an element of $C^p(G)'$ if and only if T can be expressed as a finite sum of derivatives of a class of functions exclusively parameterized by members of $\mathfrak{F}^\epsilon$ and $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$ (with $\epsilon = (\frac{2}{p}) - 1$ for all $1 \leq p \leq 2$). This gives the non-spherical abstract version of the extension theorem for any positive-definite distribution on G . Our results confirm the one-to-one correspondence between tempered invariant positive-definite distributions and the Bochner measures of the case $SU(1,1)/\{\pm 1\}$ for all G .

Subject Classification: 43A85, 22E30, 22E46

Keywords: Spherical Bochner theorem: Tempered invariant distributions: Harish-Chandra's Schwartz algebras

§1. Introduction.

The classical *Bochner theorem* is one of the cornerstones of harmonic analysis on $\mathbb{R}^n$ and it states that every *positive-definite distribution* on $\mathbb{R}^n$ is the *Fourier transform* of some unique *tempered measure* on $\mathbb{R}^n$. The proof of this theorem depends on being able to express such a distribution as a finite sum of derivatives of bounded functions on $\mathbb{R}^n$ and showing that the distribution is tempered. Underlying this proof is the use of the Fourier transform theory of *Schwartz space* on $\mathbb{R}^n$ which makes it possible to, in the first instance, define a distribution on $\mathbb{R}^n$.

In generalizing the classical Bochner theorem to other *topological groups* W. H. Barker showed that every positive-definite distribution on a *unimodular Lie group* may be expressed as a finite sum of derivatives of bounded functions. However, the Fourier transform theory for Schwartz space is not available for unimodular Lie groups neither had the theory been completely successful for specific classes of unimodular Lie groups, except for isolated examples like $SL(2, \mathbb{R})$. The most successful Fourier transform theory for a class of unimodular Lie groups had been the *Trombi-Varadarajan theory* [14.] constructed only on the *spherical* part of a connected semisimple Lie group G and this informed the consideration of the *spherical Bochner theorem* (which still lacks the classical temperedness) and its *spherical extension* Baker [3.] (which may be consulted for other partial versions and references).

In this paper, we shall however employ the recently announced results of Oyadare on the full scalar-valued transform theory for G to give the *full scalar-valued Bochner theorem*. The paper is organized as follows. Section 2 contains preliminary matters on the *structure theory* of G , explicit parametrization of its *spherical functions* and the *Helgason-Johnson theorem*. The results leading to the spherical Bochner theorem and spherical extension are discussed in section 3, where the *Fundamental Theorem of Harmonic Analysis on G* (which generalizes the Trombi-Varadarajan theorem) is stated as Theorem 3.2 (*cf.* [13.]). The last section contain the main contributions of this paper, where the existence of the Fundamental Theorem paves way to the uplifting of Barker's spherical results to all of G (Theorems 4.2 and 4.3). An *abstract extension* (Conjecture 4.5) of our results suggests that the starting point of expressing a positive-definite distribution as a finite sum of derivatives of bounded functions may need to be critically looked at in order to recover their classical temperedness.

§2. Structure of the group G and its spherical functions.

Let G be a connected semisimple Lie group with finite center, we denote its Lie algebra by $\mathfrak{g}$ whose *Cartan decomposition* is given as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Denote by θ the *Cartan involution* on $\mathfrak{g}$ whose collection of fixed points is $\mathfrak{t}$. We also denote by K the analytic subgroup of G with Lie algebra $\mathfrak{t}$. K is then a maximal compact subgroup of G . Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ with algebraic dual $\mathfrak{a}^*$ and set $A = \exp \mathfrak{a}$. For every $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

and call λ a restricted root of $(\mathfrak{g}, \mathfrak{a})$ whenever $\mathfrak{g}_\lambda \neq \{0\}$. Denote by $\mathfrak{a}'$ the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$, and call its connected components the *Weyl chambers*. Let $\mathfrak{a}^+$ be one of the Weyl chambers, define the restricted root λ positive whenever it is positive on $\mathfrak{a}^+$ and denote by Δ^+ the set of all restricted positive roots. Members of Δ^+ which form a basis for Δ and can not be written as a linear combination of other members of Δ^+ are called *simple*. We then have the *Iwasawa decomposition* $G = KAN$, where N is the analytic subgroup of G corresponding to $\mathfrak{n} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$, and the *polar decomposition* $G = K \cdot cl(A^+) \cdot K$, with $A^+ = \exp \mathfrak{a}^+$, and $cl(A^+)$ denoting the closure of A^+ .

If we set $M = \{k \in K : Ad(k)H = H, H \in \mathfrak{a}\}$ and $M' = \{k \in K : Ad(k)\mathfrak{a} \subset \mathfrak{a}\}$ and call them the *centralizer* and *normalizer* of $\mathfrak{a}$ in K , respectively, then (see [9], p. 284); (i) M and M' are compact and have the same Lie algebra and (ii) the factor $\mathfrak{w} = M'/M$ is a finite group called the *Weyl group*. $\mathfrak{w}$ acts on $\mathfrak{a}_{\mathbb{C}}^*$ as a group of linear transformations by the requirement

$$(s\lambda)(H) = \lambda(s^{-1}H),$$

$H \in \mathfrak{a}$, $s \in \mathfrak{w}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the complexification of $\mathfrak{a}^*$. We then have the *Bruhat decomposition*

$$G = \bigsqcup_{s \in \mathfrak{w}} Bm_sB$$

where $B = MAN$ is a closed subgroup of G and $m_s \in M'$ is the representative of s (i.e., $s = m_sM$). The Weyl group invariant members of a space shall be denoted by the superscript $^{\mathfrak{w}}$.

Some of the most important functions on G are the *spherical functions* which we now discuss as follows. A non-zero continuous function φ on G

shall be called a (*zonal*) *spherical function* whenever $\varphi(e) = 1$, $\varphi \in C(G//K)$ where

$$C(G//K) := \{g \in C(G) : g(k_1 x k_2) = g(x), \ k_1, k_2 \in K, \ x \in G\}$$

and $f * \varphi = (f * \varphi)(e) \cdot \varphi$ for every $f \in C_c(G//K)$, where

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)dy.$$

This leads to the existence of a homomorphism $\lambda : C_c(G//K) \rightarrow \mathbb{C}$ given as $\lambda(f) = (f * \varphi)(e)$. This definition is equivalent to the satisfaction of the functional relation

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y), \quad x, y \in G.$$

It has been shown by Harish-Chandra [8.] that spherical functions on G can be parametrized by members of $\mathfrak{a}_{\mathbb{C}}^*$. Indeed every spherical function on G is of the form

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - \rho)H(xk)} dk, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

$\rho = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda$, where $m_\lambda = \dim(\mathfrak{g}_\lambda)$, and that $\varphi_\lambda = \varphi_\mu$ iff $\lambda = s\mu$ for some $s \in \mathfrak{w}$. Some of the well-known properties of spherical functions are $\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$, $\varphi_{-\lambda}(x) = \bar{\varphi}_\lambda(x)$, $|\varphi_\lambda(x)| \leq \varphi_{\Re \lambda}(x)$, $|\varphi_\lambda(x)| \leq \varphi_{i\Im \lambda}(x)$, $\varphi_{-i\rho}(x) = 1$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, while $|\varphi_\lambda(x)| \leq \varphi_0(x)$, $\lambda \in i\mathfrak{a}^*$, $x \in G$. Also if Ω is the *Casimir operator* on G then

$$\Omega \varphi_\lambda = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle) \varphi_\lambda,$$

where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\langle \lambda, \mu \rangle := \text{tr}(adH_\lambda adH_\mu)$ for elements $H_\lambda, H_\mu \in \mathfrak{a}$. The elements $H_\lambda, H_\mu \in \mathfrak{a}$ are uniquely defined by the requirement that $\lambda(H) = \text{tr}(adH adH_\lambda)$ and $\mu(H) = \text{tr}(adH adH_\mu)$ for every $H \in \mathfrak{a}$ ([9.], Theorem 4.2). Clearly $\Omega \varphi_0 = 0$.

Due to a hint dropped by Dixmier [6.] (*cf.* [12.]) in his discussion of some functional calculus, it is necessary to recall the notion of a '*positive-definite*' function and then discuss the situation for positive-definite spherical functions. We call a continuous function $f : G \rightarrow \mathbb{C}$ (algebraically) positive-definite whenever, for all $x_1, \dots, x_m$ in G and all $\alpha_1, \dots, \alpha_m$ in $\mathbb{C}$, we have

$$\sum_{i,j=1}^m \alpha_i \bar{\alpha}_j f(x_i^{-1} x_j) \geq 0.$$

It can be shown (cf. [9.]) that $f(e) \geq 0$ and $|f(x)| \leq f(e)$ for every $x \in G$ implying that the space $\wp$ of all positive-definite spherical functions on G is a subset of the space $\mathfrak{F}^1$ of all bounded spherical functions on G .

We know, by the Helgason-Johnson theorem ([11.]), that

$$\mathfrak{F}^1 = \mathfrak{a}^* + iC_\rho$$

where C_ρ is the convex hull of $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$. Defining the *involution* f^* of f as $f^*(x) = f(x^{-1})$, it follows that $f = f^*$ for every $f \in \wp$, and if $\varphi_\lambda \in \wp$, then λ and $\bar{\lambda}$ are Weyl group conjugate, leading to a realization of $\wp$ as a subset of $\mathfrak{w} \setminus \mathfrak{a}_{\mathbb{C}}^*$. $\wp$ becomes a locally compact Hausdorff space when endowed with the *weak*-topology* as a subset of $L^\infty(G)$.

We denote the set of equivalence classes of the necessarily finite-dimensional irreducible representations of K by $\mathcal{E}(K)$ whose character is $\chi_{\mathfrak{d}}$, for every $\mathfrak{d} \in \mathcal{E}(K)$. The class functions $\xi_{\mathfrak{d}} : K \rightarrow \mathbb{C}$ defined as $\xi_{\mathfrak{d}}(k) := \dim(\mathfrak{d})\chi_{\mathfrak{d}}(k^{-1})$ are idempotents (i.e., $\xi_{\mathfrak{d}} * \xi_{\mathfrak{d}} = \xi_{\mathfrak{d}}$ with $\xi_{\mathfrak{d}_1} * \xi_{\mathfrak{d}_2} = 0$ whenever $\mathfrak{d}_1 \neq \mathfrak{d}_2$). Choosing π to be any representation of K (which may be the restriction to K of a representation of G) in a complete Hausdorff locally convex space, V , a continuous projection operator on V may be given as the image of $\xi_{\mathfrak{d}}$ under π . That is,

$$E_{\pi, \mathfrak{d}} := \pi(\xi_{\mathfrak{d}}) = \int_K \xi_{\mathfrak{d}}(k)\pi(k)dk = \dim(\mathfrak{d}) \int_K \chi_{\mathfrak{d}}(k^{-1})\pi(k)dk$$

(Here $\int_K dk = 1$) Idempotency of $\xi_{\mathfrak{d}}$ assures that $E_{\pi, \mathfrak{d}}$ is indeed a projection on V (since $E_{\pi, \mathfrak{d}}^2 = E_{\pi, \mathfrak{d}}$ and $E_{\pi, \mathfrak{d}_1}E_{\pi, \mathfrak{d}_2} = 0$ whenever $\mathfrak{d}_1 \neq \mathfrak{d}_2$) and that its range, written as $V_{\mathfrak{d}} (= E_{\pi, \mathfrak{d}}(V))$ is a closed linear subspace of V consisting mainly of members of V which *transform according to* $\mathfrak{d}$. See Oyadare [13.], p. 109.

The closed linear subspace $V_{\mathfrak{d}}$ above becomes familiar when $V = \mathcal{C}^p(G)$ under the usual regular representation. In this case the left and right regular representations are denoted as l and r given as $(l(x)f)(y) = f(x^{-1}y)$ and $(r(x)f)(y) = f(yx)$, respectively; $x, y \in G$, $f \in \mathcal{C}^p(G)$; and it may be computed that for any $\mathfrak{d} \in \mathcal{E}(K)$, $E_{l, \mathfrak{d}} = l(\xi_{\mathfrak{d}})$ and $E_{r, \mathfrak{d}} = r(\xi_{\mathfrak{d}})$ are the respective operators of left and right convolutions by the measure $\xi_{\mathfrak{d}}dk$ and $\tilde{\xi}_{\mathfrak{d}}dk = \xi_{\bar{\mathfrak{d}}}dk$, respectively. Here $\bar{\mathfrak{d}}$ is the class contragredient to $\mathfrak{d}$. We therefore have a representation $l \times r$ of $G \times G$ on $\mathcal{C}(G)$ given as $((l \times r)(x, y)f)(z) = f(x^{-1}yz)$, $x, y, z \in G$ and the corresponding projection $E_{l \times r, (\mathfrak{d}_1 \times \mathfrak{d}_2)} = (l \times r)(\xi_{(\mathfrak{d}_1 \times \mathfrak{d}_2)})$, which from the above remarks could be computed as

$$E_{l \times r, (\mathfrak{d}_1 \times \mathfrak{d}_2)}f = \xi_{\mathfrak{d}_1} * f * \xi_{\mathfrak{d}_2},$$

$f \in \mathcal{C}^p(G)$.

We now choose $\mathfrak{d} \in \mathcal{E}(K)$. The image of $\mathcal{C}^p(G)$ under $E_{l \times r, (\mathfrak{d} \times \bar{\mathfrak{d}})}$ is the closed subalgebra of $\mathcal{C}^p(G)$ denoted as $\mathcal{C}_{\mathfrak{d}}^p(G)$ and is exactly given as

$$\mathcal{C}_{\mathfrak{d}}^p(G) = \xi_{\mathfrak{d}} * \mathcal{C}^p(G) * \xi_{\mathfrak{d}} = \{\xi_{\mathfrak{d}} * f * \xi_{\mathfrak{d}} : f \in \mathcal{C}^p(G)\}.$$

Thus the members of $\mathcal{C}_{\mathfrak{d}}^p(G)$ are those of $\mathcal{C}^p(G)$ which may be written as $\xi_{\mathfrak{d}} * f * \xi_{\mathfrak{d}}$ for some $f \in \mathcal{C}^p(G)$. That is, every $g \in \mathcal{C}_{\mathfrak{d}}^p(G)$ is given as $g = \xi_{\mathfrak{d}} * f * \xi_{\mathfrak{d}}$, with $f \in \mathcal{C}^p(G)$. We shall henceforth write members g of $\mathcal{C}_{\mathfrak{d}}^p(G)$ as $g_{\mathfrak{d}, f}$.

§3. The Spherical Bochner Theorem and Extension

Let

$$\varphi_0(x) := \int_K \exp(-\rho(H(xk))) dk$$

be denoted as $\Xi(x)$ and define $\sigma : G \rightarrow \mathbb{C}$ as

$$\sigma(x) = \|X\|$$

for every $x = k \exp X \in G$, $k \in K$, $X \in \mathfrak{a}$, where $\|\cdot\|$ is a norm on the finite-dimensional space $\mathfrak{a}$. These two functions are spherical functions on G and there exist numbers c, d such that

$$1 \leq \Xi(a) e^{\rho(\log a)} \leq c(1 + \sigma(a))^d.$$

Also there exists $r_0 > 0$ such that $c_0 =: \int_G \Xi(x)^2 (1 + \sigma(x))^{r_0} dx < \infty$ (Varadarajan [16.], p. 231). For each $0 \leq p \leq 2$ define $\mathcal{C}^p(G)$ to be the set consisting of functions f in $C^\infty(G)$ for which

$$\|f\|_{g_1, g_2; m} := \sup_G |f(g_1; x; g_2)| \Xi(x)^{-2/p} (1 + \sigma(x))^m < \infty$$

where $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, the *universal enveloping algebra* of $\mathfrak{g}_{\mathbb{C}}$, $m \in \mathbb{Z}^+$, $x \in G$, $f(x; g_2) := \left. \frac{d}{dt} \right|_{t=0} f(x \cdot (\exp t g_2))$ and $f(g_1; x) := \left. \frac{d}{dt} \right|_{t=0} f((\exp t g_1) \cdot x)$. We call $\mathcal{C}^p(G)$ the Schwartz space on G for each $0 < p \leq 2$ and note that $\mathcal{C}^2(G)$ is the well-known (See Arthur [1.] and [2.]) Harish-Chandra space of rapidly decreasing functions on G . The inclusions

$$C_c^\infty(G) \subset \mathcal{C}^p(G) \subset L^p(G)$$

hold and with dense images. It also follows that $\mathcal{C}^p(G) \subseteq \mathcal{C}^q(G)$ whenever $0 \leq p \leq q \leq 2$. Each $\mathcal{C}^p(G)$ is closed under *involution* and the *convolution*,

*. Indeed $\mathcal{C}^p(G)$ is a Fréchet algebra (Varadarajan [15.], p. 69). We endow $\mathcal{C}^p(G//K)$ with the relative topology as a subset of $\mathcal{C}^p(G)$.

For any measurable function f on G we define the *Harish-Chandra Fourier transform* $\widehat{f}$ as

$$\widehat{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) dx,$$

$\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We shall call it *spherical* whenever f is K -biinvariant and call it *scalar-valued* whenever $f \in G$. It is known (See Baker [3.]) that for $f, g \in L^1(G)$ we have:

(i.) $(f * g)^\wedge = \widehat{f} \cdot \widehat{g}$ on $\mathfrak{F}^1$ whenever f (or g) is right - (or left-) K -invariant;

(ii.) $(f^*)^\wedge(\varphi) = \overline{\widehat{f}(\varphi^*)}$, $\varphi \in \mathfrak{F}^1$; hence $(f^*)^\wedge = \overline{\widehat{f}}$ on $\wp$: and, if we define

$$f^\#(g) := \int_{K \times K} f(k_1 x k_2) dk_1 dk_2, x \in G,$$

then

(iii.) $(f^\#)^\wedge = \widehat{f}$ on $\mathfrak{F}^1$.

In order to know the image of the spherical Fourier transform when restricted to $\mathcal{C}^p(G//K)$ we need the following spaces that are central to the statement of the well-known result of Trombi and Varadarajan [14.] (See Theorem 3.1 below).

Let C_ρ be the closed convex hull of the (finite) set $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$, i.e.,

$$C_\rho = \left\{ \sum_{i=1}^n \lambda_i (s_i \rho) : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, s_i \in \mathfrak{w} \right\}$$

where we recall that, for every $H \in \mathfrak{a}$,

$$(s\rho)(H) = \frac{1}{2} \sum_{\lambda \in \Delta^+} m_\lambda \cdot \lambda(s^{-1}H).$$

Now for each $\epsilon > 0$ set $\mathfrak{F}^\epsilon = \mathfrak{a}^* + i\epsilon C_\rho$. Each $\mathfrak{F}^\epsilon$ is convex in $\mathfrak{a}_{\mathbb{C}}^*$ and

$$\text{int}(\mathfrak{F}^\epsilon) = \bigcup_{0 < \epsilon' < \epsilon} \mathfrak{F}^{\epsilon'}$$

(Trombi and Varadarajan [14.], Lemma (3.2.2)). Let us define $\mathcal{Z}(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and, for each $\epsilon > 0$, let $\mathcal{Z}(\mathfrak{F}^\epsilon)$ be the space of all $\mathbb{C}$ -valued functions Φ such that (i.) Φ is defined and holomorphic on $\text{int}(\mathfrak{F}^\epsilon)$, and (ii.) for each holomorphic differential operator D with polynomial coefficients we have $\sup_{\text{int}(\mathfrak{F}^\epsilon)} |D\Phi| < \infty$. The space $\mathcal{Z}(\mathfrak{F}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)}$, of seminorms given by $\|\Phi\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)} := \sup_{\text{int}(\mathfrak{F}^\epsilon)} |D\Phi|$. It is known that $D\Phi$ above extends to a continuous function on all of $\mathfrak{F}^\epsilon$ (Trombi and Varadarajan [14.], pp. 278–279). An appropriate subalgebra of $\mathcal{Z}(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$ -invariant elements of $\mathcal{Z}(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. The following well-known result affords us the opportunity of defining a distribution on $\mathcal{C}^p(G//K)$.

3.1 Theorem (Trombi-Varadarajan [14.]). *Let $0 < p \leq 2$ and set $\epsilon = (\frac{2}{p}) - 1$. Then the spherical Harish-Chandra Fourier transform $f \mapsto \hat{f}$ is a linear topological algebra isomorphism of $\mathcal{C}^p(G//K)$ onto $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. $\square$*

It has been shown by Oyadare [13.] that the function $(\hat{\xi}_0)^{-1}$ exists and lives on $K/M \times \text{int}(\mathfrak{F}^\epsilon)$ and if we define $\mathcal{C}^p(\hat{G})$ as

$$\mathcal{C}^p(\hat{G}) = \{(\hat{\xi}_1)^{-1} \cdot h \cdot (\hat{\xi}_1)^{-1} : h \in \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)\}$$

for $0 < p \leq 2$, then a complete generalization of Theorem 3.1 to all of $\mathcal{C}^p(G)$ is possible, allowing us define a distribution on $\mathcal{C}^p(G)$, and is given as follows.

3.2. Theorem (The Fundamental Theorem of Harmonic Analysis on G [13.]). *Let $0 < p \leq 2$, then the full scalar-valued Harish-Chandra Fourier transform sets up a linear topological algebra isomorphism $\mathcal{C}^p(G) \rightarrow \mathcal{C}^p(\hat{G})$. $\square$*

In order to use the above theorems to re-state and generalize the results of Barker [3.], we require the following notions.

3.3 Definitions.

- (i.) A distribution S on G (i.e., $S \in C_c^\infty(G)'$) is said to be (integrally) positive-definite (written as $S \gg 0$) whenever

$$S[f * f^*] \geq 0,$$

for $f \in C_c^\infty(G)$.

- (ii.) A distribution S on G is called K -bi-invariant whenever $S^\# = S$ where

$$S^\#[f] := S[f^{L(k_1)R(k_2)}],$$

for $f \in C_c^\infty(G)$.

- (iii.) A measure μ defined on $\wp$ is said to be of polynomial growth if there exists a holomorphic polynomial Q on $\mathfrak{a}_{\mathbb{C}}^*$ such that $\int_{\wp} (d\mu/|Q|) < \infty$.
- (iv.) The support, $\text{supp}(\mu)$, of a regular Borel measure μ is the smallest closed set A such that $\mu(B) = 0$ for all Borel sets B disjoint from A .

One of the two well-known most general results on positive-definite distributions on G (indeed, on any unimodular Lie group) and which started off investigations leading to Bochner theorem (on $\mathbb{R}^n$) is the following generalization by Barker [4.] used in the proof of his Bochner theorems.

3.4 Theorem Every $S \in (C_c^\infty(G))'$ in which $S \gg 0$ can be expressed as the finite sum of

$$S = \sum_j D^j E^j f_j$$

where each f_j is a bounded function on G and D^j (respectively, E^j) is a left (respectively, right) invariant differential operators. $\square$

The following is the first of the main results of Baker [3.] giving the spherical Bochner theorem (cf. [8.]).

3.5 Theorem (The spherical Bochner theorem) Let $S \in (C_c^\infty(G))'$ in which $S \gg 0$. Then S extends uniquely to an element in $(C^1(G))'$ and there exists a unique positive regular Borel measure μ of polynomial growth on $\wp$ such that

$$S[f] = \int_{\wp} \hat{f} d\mu, \quad f \in C^1(G//K).$$

The correspondence between S and μ is bijective when restricted to K -bi-invariant distributions, in which case the formula holds for all $f \in C^1(G)$. $\square$

The unique Borel measure μ in the last theorem shall be called the (spherical) Bochner measure of T . The second of the main results of Baker [3.] (which relates the largest $C^p(G//K)$ -space to which the positive-definite distribution S of Theorem 3.4 can be extended with the support of its (spherical) Bochner measure) is also a consequence of the Trombi-Varadarajan theorem (Theorem 3.1 above) and is stated as follows.

3.6 Theorem (The extension theorem) Suppose S is a positive-definite distribution with spherical Bochner measure μ . Then $S \in (C^p(G//K))'$ if and only if $\text{supp}(\mu) \subset \mathfrak{F}^\epsilon$ where $1 \leq p \leq 2$ and $\epsilon = (\frac{2}{p}) - 1$. In such a case

$$S[f] = \int_{\wp} \hat{f} d\mu, \quad f \in C^p(G//K). \quad \square$$

It is our modest aim in the next section to show that Barker [3.] had the needed framework to arrive at the full Bochner theorem on G , but was hampered by the non-availability of the full theory for the Harish-Chandra Fourier transform on $\mathcal{C}^p(G)$ (given here as Theorem 3.2) and was therefore restricted to the spherical Bochner theorem derived through Theorem 3.1. It is now possible, in the light of Theorem 3.2 to investigate the full Bochner theorem as done in the next section.

§4. The full Bochner theorem.

We shall in the present section, establish the full Bochner theorem effortlessly from the results of §3. It will be clear from our results that the spherical Bochner measure corresponding to a positive-definite distribution on G (of Theorems 3.5 and 3.6 above) is still the positive regular Borel measure for each $T \in (\mathcal{C}^p(G))'$. An indication of this fact is however already contained in the first part of Theorem 3.5. Even though the (so-called *spherical*) Bochner measure is still defined in this section on $\wp$ (indeed, $\text{supp } (\mu) = \wp$), we shall henceforth refer to it simply as *the Bochner measure*. This is partly due to the main result of Oyadare [13] (which is Theorem 3.2 above) which shows that the Trombi-Varadarajan spherical image $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ is indeed the *nucleus* of the image of the full Harish-Chandra Fourier transform on all of G and partly due to the next theorem.

We start by recalling a continuous surjection map of $\mathcal{C}^p(G)$ onto $\mathcal{C}^p(G//K)$.

4.1 Lemma. *Let $\varphi \in C_c^\infty(G)$ and consider $\varphi^\# \in C_c^\infty(G//K)$. Then $\varphi \mapsto \varphi^\#$ is a continuous linear map of $\mathcal{C}^p(G)$ onto $\mathcal{C}^p(G//K)$.*

Proof. $\varphi \mapsto \varphi^\#$ is a continuous linear map of $C_c^\infty(G)$ onto $C_c^\infty(G//K)$, which extends to $\mathcal{C}^p(G)$ due to the denseness of $C_c^\infty(G)$ in $\mathcal{C}^p(G)$. $\square$

Indeed, $\varphi \mapsto \varphi^\#$ is a continuous endomorphism of $\mathcal{C}^p(G)$ onto $\mathcal{C}^p(G//K)$, as proved in Proposition 3.2 of [3.]. The following is our first main result.

4.2 Theorem (The full Bochner theorem) *Let $T \in (C_c^\infty(G))'$ in which $T \gg 0$. Then T extends uniquely to an element in $(\mathcal{C}^1(G))'$ and there exists a unique positive regular Borel measure μ of polynomial growth on $\wp$ such that*

$$T[f] = \int_{\wp} \hat{f} d\mu, \quad f \in \mathcal{C}^1(G).$$

The correspondence between T and μ is bijective when restricted to K -bi-invariant distributions.

Proof. Assertion (i.) follows from the first part of Theorem 3.5. For (ii.) let $f \in \mathcal{C}^1(G)$. Then by Lemma 4.1, $f \mapsto f^\#$ is a continuous linear map of $\mathcal{C}^1(G)$ onto $\mathcal{C}^1(G//K)$ and $\widehat{(f^\#)} = \widehat{f}$ on $\mathfrak{F}^1$, hence on $\wp$ (by Helgason-Johnson theorem). This means that the spherical Harish-Chandra Fourier transform of the spherical function $f^\#$ is exactly the full Harish-Chandra Fourier transform of the function f on G .

Now, since $T \in (\mathcal{C}^1(G))'$, it follows from Proposition 3.2 of Baker [3.] that $T|_{\mathcal{C}^1(G//K)} \in (\mathcal{C}^1(G//K))'$. We shall denote $T|_{\mathcal{C}^1(G//K)}$ by S_T and note that it is such that $T[f] = S_T[f^\#]$ for all $f \in \mathcal{C}^1(G)$. It follows from Theorem 3.5 therefore that

$$T[f] = S_T[f^\#] = \int_{\wp} \widehat{(f^\#)} d\mu = \int_{\wp} \widehat{f} d\mu,$$

as required for all $f \in \mathcal{C}^1(G)$. The claim in (iii.) holds by viewing the bijective correspondence of Theorem 3.5 in the light of (ii.). $\square$

The unique positive regular Borel measure μ in Theorem 4.2 shall simply be called the Bochner measure of T . In the light of Theorem 4.2 we shall now prove the full version of Barker's Extension (Theorem 3.6) of the spherical Bochner theorem as follows. This will be our first extension of Theorem 4.2.

4.3 Theorem (First extension theorem) *Suppose T is a positive-definite distribution with Bochner measure μ . Then $T \in (\mathcal{C}^p(G))'$ if and only if $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$ where $1 \leq p \leq 2$ and $\epsilon = (\frac{2}{p}) - 1$. In this case*

$$T[f] = \int_{\wp} \widehat{f} d\mu, \quad f \in \mathcal{C}^p(G).$$

Proof. Given that $T \in (\mathcal{C}^p(G))'$ satisfies $T \gg 0$, then S_T defined as $S_T := T|_{\mathcal{C}^p(G//K)}$ satisfies $S_T \in (\mathcal{C}^p(G//K))'$ with $S \gg 0$, corresponding to the (spherical) Bochner measure μ . It follows therefore from Theorem 3.6 that $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$.

Conversely, let μ denote the Bochner measure of a positive-definite distribution S on G with $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$. The by Theorem 3.6, $S \in (\mathcal{C}^p(G//K))'$ and $S[f] = \int_{\wp} \widehat{f} d\mu$, for all $f \in \mathcal{C}^p(G//K)$. If we now define T as $T[f] := S[f^\#]$ for all $f \in \mathcal{C}^p(G)$, then; (i.) T is a distribution on G ; (ii.) T is positive-definite with Bochner measure μ , since $T[f * f^*] = S[(f * f^*)^\#] = \int_{\wp} \widehat{(f * f^*)^\#} d\mu = \int_{\wp} \widehat{(f * f^*)} d\mu = \int_{\wp} |\widehat{f}|^2 d\mu \geq 0$; and (iii.) for all $f \in \mathcal{C}^p(G)$, $T[f] := S[f^\#] = \int_{\wp} \widehat{(f^\#)} d\mu = \int_{\wp} \widehat{f} d\mu$. $\square$

4.4 Remarks. (i.) With $p = 2$ Theorem 4.3 establishes Theorem 9.3 of Baker [5.] for $G = SU(1, 1)/\{\pm 1\}$ for the scalar-valued full Harish-Chandra Fourier transform.

(ii.) The above theorem gives a characterization of any given p -tempered positive-definite distribution in terms of which of the tubular domains, $\mathfrak{F}^\epsilon$, contains the support of its corresponding Bochner measure, μ . The existence and explicit realization for $p = 1$ is given in Theorem 4.2, where $\text{supp } (\mu) = \wp \subset \mathfrak{F}^1$. Here and by the Helgason-Johnson theorem, $\mathfrak{F}^1$ is the exclusive indexing set for all *bounded* spherical functions, in terms of which every positive-definite distribution on G is written, according to Theorem 3.4. It is therefore safe to say that, unless Theorem 3.4 is generalized, the first extension theorem (Theorem 4.3) above may not be fully realized, as (in any analysis based on Theorem 3.4) we shall always be taken back to the nature of the functions f_j as being bounded functions; thus $\mathfrak{F}^1$ may equally surface again and leading eventually to $\mathcal{C}^1(G)$.

Indeed any change in the nature of f_j (in Theorem 3.4) is bound to reflect in the type of Bochner theorem (and extension) that would be available for consideration. A general look at the restriction introduced through Theorem 3.4 (though a very general result in its own right) is expected to follow the following line of thought.

4.5 Conjecture (Abstract extension theorem). *Let T denote a positive-definite distribution on G with Bochner measure μ and set $\epsilon = (\frac{2}{p}) - 1$ for all $1 \leq p \leq 2$. Then $T \in (\mathcal{C}^p(G))'$ if and only if T can be expressed as a finite sum of derivatives of functions that are exclusively indexed by $\mathfrak{F}^\epsilon$ with $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$. $\square$*

The proof and application of this conjecture may be useful in rectifying the anomaly of the first parts of Theorems 3.5 and 4.2, by showing that every positive-definite distribution is 2-tempered (as known for $\mathbb{R}^n$). That is, that the support of the Bochner measure is always in $\mathfrak{F}^0$.

The following is a consequence of Theorem 4.3 and the fundamental theorem (Theorem 3.2).

4.6 Corollary. *Let $T \in (\mathcal{C}^p(G))'$, $T \gg 0$, with Bochner measure μ and set $\epsilon = (\frac{2}{p}) - 1$ for all $1 \leq p \leq 2$ such that $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$. Then, for all $f \in \mathcal{C}^p(G)$,*

$$T[f] = \int_{\wp} (\hat{\xi}_1)^{-1} \cdot \hat{\varphi} \cdot (\hat{\xi}_1)^{-1} d\mu,$$

where $\varphi = f^\#$. $\square$.

§5. Conclusion. We may now embark on an application of Theorem 4.3 to p -tempered invariant positive-definite distributions in the fashion of Baker [5.]. This is possible now that, according to Theorem 4.3, every p -tempered positive-definite distribution on G is the (full) scalar-valued Harish-Chandra Fourier transform of some unique tempered measure μ (of course, with $\text{supp } (\mu) \subset \mathfrak{F}^\epsilon$). With the (full) scalar-valued Harish-Chandra Fourier transform, it will be possible to then bring the full Representation Theory of G to bear on more explicit expression for and decomposition of $T[f]$, for all $f \in \mathcal{C}^p(G)$, as is known in Baker [5.].

The operator-valued version of the fundamental theorem of harmonic analysis on G and its Bochner theorem would be the subject of another paper.

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SPECIAL NILPOTENT LIE ALGEBRAS

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Abstract

The classification of Nilpotent Lie algebras is an open and difficult problem.
The aim of the present work is to give some classes of Nilpotent Lie algebras
using Kac-Moody Lie algebras.

Key words : Nilpotent Lie algebras Kac-Moody algebras, root system,
Simple Lie algebra and Nilpotent Lie algebra of maximal rank.

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1.Introduction

Let g be a Nilpotent lie algebra over the field K of charateristic 0.We consider $K=\mathbb{C}$. The problem of classification of g is open. There are different methods to approach this problem. The aim of the present paper is to determine some Nilpotent Lie algebras of maximal rank.

The whole paper contains four paragraphs,each of them is analysed as follows.

The first paragraph is the introduction.

Some general notions and results of Kac-Moody Lie algebras are given in the second paragraph. This theory of Kac Moody Lie algebras is used for this construction of Nilpotent Lie algebras of maximal rank.

The third paragraph includes the theory of root system for a Nilpotent Lie algebra of maximal rank. The root system is a basic tool for this construction.

Some constructions of Nilpotent Lie algebras of maximal rank are given in the last paragraph.

2.KAC - MOODY LIE ALGEBRAS

Definition 2.1: Let $A(a_{ij})_{1 \leq i, j \leq \ell}$ be a square matrix with entries in $\mathbb{Z}$ with the following properties,

$$\left. \begin{array}{l} \text{(i)} \quad a_{ii} = 2 \\ \text{(ii)} \quad a_{ij} \leq 0 \text{ if } i \neq j \\ \text{(iii)} \text{ if } a_{ij} = 0 \text{ then } a_{ji} = 0, i \neq j \end{array} \right\} \text{ for } i, j = 1, \dots, \ell$$

Then the square matrix A is called *Generalized Cartan Matrix* (denoted) G.C.M.

Definition 2.2: Two G.C.M's $A(a_{ij})_{1 \leq i, j \leq \ell}$ and $B(b_{ij})_{1 \leq i, j \leq \ell}$ are called *equivalent* if $\exists \sigma \in \mathcal{G}_\ell$ (The group of the permutation of $\{1, \dots, \ell\}$) such that

$$b_{ij} = a_{\sigma i \sigma j} \quad \forall i, j = 1, \dots, \ell$$

Definition 2.3: Let L be a lie algebra with 3ℓ generators x_i, h_i, y_i ($i = 1, \dots, \ell$) and A be a G.C.M. associated to the Lie algebra L , satisfying the relations:

$$\left. \begin{array}{l} \text{(i)} \quad [h_i, h_j] = 0, \quad [x_i, y_j] = \delta_{ij} h_i \\ \text{(ii)} \quad [h_i, x_j] = a_{ij} x_j, \quad [h_i, y_j] = -a_{ij} y_j \\ \text{(iii)} \quad (\text{ad} x_i)^{1-a_{ij}} x_j = 0, \quad (\text{ad} y_i)^{1-a_{ij}} y_j = 0 \end{array} \right\} \quad i, j = 1, \dots, \ell$$

Then the Lie algebra L is called, Kac - Moody lie algebra

Definition 2.4: let $\{\alpha_1, \dots, \alpha_\ell\}$ be the canonical basis of Z^ℓ . For $\alpha \in N^\ell - \{0\}$ we have $\alpha = \sum_{i=1}^{\ell} d_i \alpha_i$. We denote by L_α (resp $L_{-\alpha}$) the subvector space of L generated by the elements $[x_{i_1}, \dots, x_{i_r}]$ (resp. $[y_{i_1}, \dots, y_{i_r}]$ where x_i (resp. y_i) appears d_i times ($[x_1, \dots, x_\ell] = [x_1[x_2 \dots x_\ell]]$). If all d_i times are not of the same sign, $L_\alpha = (0)$.

$$[2],[3] \quad L_0 = h = Kh_1 \oplus \dots \oplus Kh_\ell = \bigoplus_{i=1}^{\ell} Kh_i$$

$$\text{positive part} \quad L_+ = \bigoplus_{\alpha \in \Delta_+} L_\alpha, \quad L_- = \bigoplus_{\alpha \in \Delta_-} L_\alpha \quad \text{negative part}$$

$$L = L_+ \oplus h \oplus L_-$$

$$\text{Where} \quad \Delta = \{\alpha \in Z^\ell / \alpha \neq 0, L_\alpha \neq 0\}$$

$$\Delta_+ = \{\alpha \in N^\ell / \alpha \neq 0, L_\alpha \neq 0\} \quad \text{positive root system}$$

$$\Delta_- = \{-\alpha; \in N^\ell / \alpha \neq 0, L_\alpha \neq 0\}$$

$$\Delta = \Delta_+ \cup \{0\} \cup \Delta_-$$

$$|\alpha| = \sum_{i=1}^{\ell} d_i \quad \text{height of a root } \alpha$$

3. ROOT SYSTEM FOR A NILPOTENT LIE ALGEBRA of MAXIMAL RANK

Definition 3.1: Let g be a Lie algebra of finite dimension. We denote by Derg and $\text{Aut}g$ be the derivation algebra and the automorphism group of g respectively.

We call a *Torus* T on g a commutative subalgebra of Derg which consists of semi-simple endomorphisms. A torus is a *maximal torus* on g , if it is not contained strictly in any other torus.

$$\begin{aligned} g : T \times g &\rightarrow g(t, x) \rightarrow tx \quad \forall t \in T, \forall x \in g. \\ T : g &= \bigoplus_{\alpha \in T^*} g_{\alpha} , \\ g_{\alpha} &= \{x \in g; tx = \alpha(t)x, \forall t \in T\} \\ \downarrow \quad \downarrow & \\ \text{root} \quad \text{root} & \\ \text{space} \quad \text{vectors} & \end{aligned}$$

The root system of g associated to T is defined by

$$\Delta = \{\alpha \in T^*; g_{\alpha} \neq (0)\}$$

If g is nilpotent, we define the *type of g* to be the $\dim(g/[g, g])$

Lemma 3.2. Let g be nilpotent Lie algebra of type ℓ and T the maximal torus on g . If $(x_1, \dots, x_{\ell})$ a T -msg and α_i the root of x_i , we have

$$\dim T = \text{rank}(\alpha_1, \dots, \alpha_{\ell}) = \text{rank} g = k,$$

and then $k \leq \ell$. If $k = \ell$, then g is of *maximal rank*.

Theorem 3.3.[1] To any Nilpotent Lie algebra of maximal rank and of type ℓ , one can associate an $\ell \times \ell$ Cartan matrix A , whose equivalence class is an invariant of g and which is characterized by the following property: To any maximal torus T and any T -msg $(x_1, \dots, x_\ell) \exists \sigma \in G_\ell: \forall i, j = 1, \dots, \ell, i \neq j$, we have

$$(\text{ad}_{x_{\sigma i}})^{-a_{ij}} x_{\sigma j} \neq 0 \quad \text{and} \quad (\text{ad}_{x_{\sigma i}})^{1-a_{ij}} x_{\sigma j} = 0$$

Theorem 3.4.[1]. Let g be a Nilpotent Lie algebra of maximal rank. Then the mapping:

$$G_\ell(g).I \rightarrow L_+(g) \bigg/ \bigoplus_{\alpha \in I} g_\alpha \quad \text{is a Lie algebra}$$

is a bijection from the set of all $G_\ell(g)$ -orbits of $I(A)$ onto Nilpotent.(max, g) (the set of all Nilpotent Lie algebras of maximal rank)

* $(I\Delta_+(g) \text{ if } \forall \alpha \in I \rightarrow \alpha + \alpha_i \in I, \forall i = 1, \dots, \ell, \alpha + \alpha_i \in \Delta_+(g))$

Theorem 3.5.[4]. Let L be a semisimple Lie algebra over an algebraically closed field of characteristic 0, and let h be a Cartan subalgebra of L . Then the root spaces for non-zero roots relative to h are *one-dimensional*. No integral multiple $i\alpha$ of a non-zero root α is again a root for $i \neq 0, \pm 1$.

Theorem 3.6.[5]. An automorphism of the Dynkin diagram is a permutation σ , of the vertices preserving both the weights and the multiplicity of the links. Equivalently, an automorphism of the Dynkin Diagram is a permutation σ , of the vertices preserving the group h (including the arrows).

Theorem 3.7.[4]. The factor group $\text{Aut}(\Delta)/W(\Delta)$ is isomorphic to the group of automorphisms of the Dynkin diagram. it is given as follows:

- (i) for $A_1, B_\ell, C_\ell, E_7, E_8, F_4, G_2$: Z_1
- (ii) for $A_\ell (\ell \geq 2), D_\ell (\ell > 4), E_6$: Z_2
- (iii) for D_4 : G_3

where Z_m is the cyclic group of order m and G_3 the permutation group of three letters.

4.NIPLTENT LIE ALGEBRAS OF MAXIMAL RANK

Construction 4.1[1]. We consider the Nilpotent Lie Algebra of dimension four g_4 , that is :

$$g_4 = kx_1 \oplus \dots \oplus kx_4$$

$$[x_1, x_2] = x_3$$

$$[x_1, x_3] = x_4$$

$$T = kt_1 \oplus kt_2 \quad ; \quad t_i x_j = \delta_{ij} x_j \quad (i, j = 1, 2)$$

$$T\text{-msg: } (x_1, x_2) \quad \text{Roots: } (\alpha_1, \alpha_2):$$

$$\alpha_1(ij) = \delta_{ij} \quad (i,j = 1,2)$$

Root space decomposition:

$$\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_{2\alpha_1 + \alpha_2}$$

$$\mathfrak{g}_{\alpha_1} = Kx_1 \quad \text{TYPE: } \ell = 2$$

$$\mathfrak{g}_{\alpha_2} = Kx_2 \quad \text{NILPOTENCY: } p = 3$$

$$\mathfrak{g}_{\alpha_1 + \alpha_2} = Kx_3 \quad \text{G.C.M: } A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} = B_2(B_1, l=2)$$

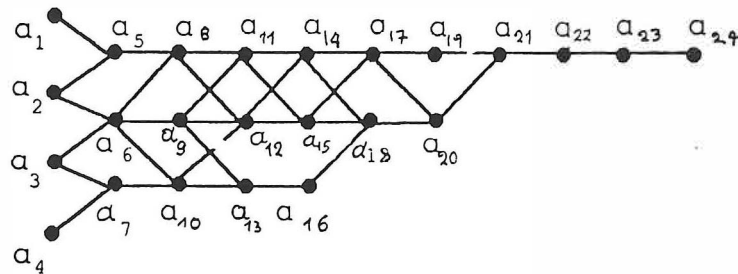
$$\mathfrak{g}_{2\alpha_1 + \alpha_2} = Kx_4 \quad \text{Dynkin Diag. } \begin{array}{cc} o & \Rightarrow o \\ 1 & 2 \end{array}$$

Conclusion: $\mathfrak{g}_4 = L_+(B_2)$

Construction 4.2.[6]. We consider the Nilpotent Lie Algebra of maximal rank end of type F_4 (exceptional). For this we have :

$$\overline{F}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \text{ Cartan matrix of } F_4$$

$$\Delta_+ = \{\alpha_1, \dots, \alpha_{24}\} \text{ whose base is: } \underbrace{\{\alpha_1, \dots, \alpha_4\}}_{\text{Simple roots}}$$



$$N = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$$

Number of Nilpotent Lie algebras of maximal rank and of type F_4 : 43 (non isomorphic)

Dynkin Diagram: $\alpha_1 \text{ --- } \alpha_2 \Rightarrow \alpha_3 \text{ --- } \alpha_4$

The highest root: $\alpha_{24} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ (fig. 11)

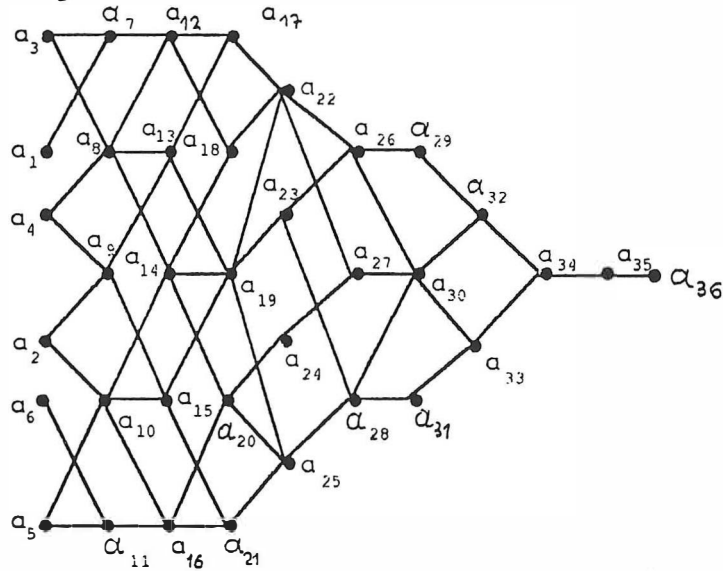
Construction 4.3.[7] Let g be a Nilpotent Lie Algebra of maximal rank end of type E_6 (exceptional). For g we obtain :

$$\bar{E}_6 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \text{ Cartan matrix of } E_6$$

Root system (positive): $\Delta_+ = \{\alpha_1, \dots, \alpha_{36}\}$ which base is: $\{\alpha_1, \dots, \alpha_6\}$

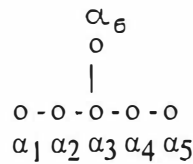
Chain: $N = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_5, \alpha_4 + \alpha_6\}$

highest root: $\alpha_{36} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
 high of $\alpha_3 = 11$



The number of Nilpotent Lie Algebras of maximal rank non isomorphic and of type E_6 is 149

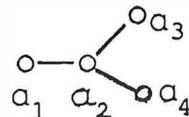
Dynkin Diagram:

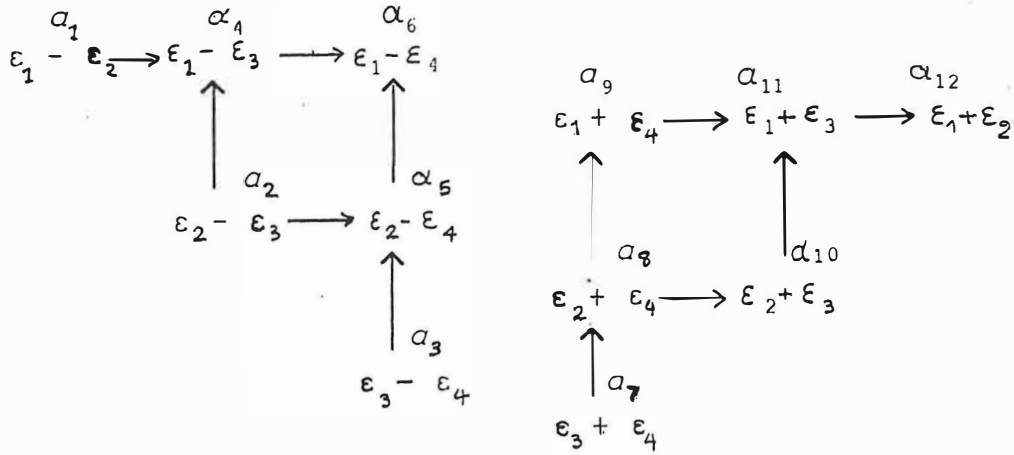


Construction 4.4.[8]. Let g be a Nilpotent Lie Algebra of maximal rank end of type D_4 . For this algebra we have :

$$\bar{D}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \text{ The Cartan matrix of } D_4$$

Dynkin diagram:





Root vectors

x_{12}	$\rightarrow x_{13}$	$\rightarrow x_{14}$		
	$\uparrow$	$\uparrow$	y_{14}	$\rightarrow y_{13} \rightarrow y_{12}$
	x_{23}	$\rightarrow x_{24}$	$\uparrow$	$\uparrow$
		$\uparrow$	y_{24}	$\rightarrow y_{23}$
		x_{34}	$\uparrow$	
			y_{34}	
$[x_{12}, x_{23}] = x_{13}$	$[x_{12}, y_{23}] = y_{13}$	$[x_{13}, y_{23}] = -y_{12}$	$[x_{23}, y_{13}] = y_{12}$	
$[x_{12}, x_{24}] = x_{14}$	$[x_{12}, y_{24}] = y_{14}$	$[x_{14}, y_{24}] = -y_{12}$	$[x_{24}, y_{14}] = y_{12}$	
$[x_{13}, x_{34}] = x_{14}$	$[x_{13}, y_{34}] = y_{14}$	$[x_{14}, y_{34}] = -y_{13}$	$[x_{34}, y_{14}] = y_{13}$	
$[x_{23}, x_{34}] = x_{24}$	$[x_{23}, y_{24}] = y_{24}$	$[x_{24}, y_{34}] = -y_{23}$	$[x_{34}, y_{24}] = y_{23}$	

The automorphism group $G_4(D_4)$ of the Dynkin diagram is the subgroup (isomorphic to G_3) of G_4 which fixes 2. It's action on Δ_+ is:

$$\sigma \left[\sum_{i=1}^4 n_i \alpha_i \right] = \sum_{i=1}^4 n_i \alpha_{\sigma i}, \quad \forall \sigma \in \mathcal{G}_4 \quad (\text{Fixes 2})$$

Theorem 4.4: There are 6 Nilpotent Lie algebras of maximal rank (non isomorphic)

for $K = \mathbb{C}$ which are the following

$$D_4^1 = Kx_{12} \oplus Kx_{23} \oplus Kx_{34} \oplus Ky_{34} \oplus Kx_{13} \oplus Kx_{24} \oplus Ky_{24}$$

$$D_4^2 = D_4^1 \oplus Kx_{14}$$

$$D_4^3 = D_4^2 \oplus Ky_{14}$$

$$D_4^4 = D_4^3 \oplus Ky_{23}$$

$$D_4^5 = D_4^4 \oplus Ky_{13}$$

$$D_4^6 = D_4^5 \oplus Ky_{12}$$

proof: We can obtain the above results using the below technics

$$\left. \begin{aligned} X_{ij} &= E_{i,j} - E_{-j,-i} \\ Y_{ij} &= E_{i,-j} - E_{j,-i} \end{aligned} \right\} 1 \leq i \leq j \leq \ell$$

Where (X_{ij}, Y_{ij}) is a base of $n_+(D)$

Then by simple computation

$$\left. \begin{aligned} g_{\varepsilon_i - \varepsilon_j} &= \mathbb{C}X_{i,j} \\ g_{\varepsilon_i + \varepsilon_j} &= \mathbb{C}Y_{i,j} \end{aligned} \right\} 1 \leq i \leq j \leq \ell$$

(only for positive part of the root space)

where also (ϵ_j) is the *base of* h^* (the dual of the Cartan subalgebra of g who is h and it's *base* is $H_i = E_{jj} - E_{-i,-i}$ ($1 \leq i \leq j \leq \ell$) knowing that E_{ij} is the matrix havin 1 in the (i,j) -position.

In the cases of the affine Kac-Moody algebras we can obtain that there exists infinite series of continous families of Nilpotent Lie algebras of maximal rank with parameters.[13],[14],[15]

An unsolvable proble till now is to classify the the Nilpotent Lie algebras of maximal rank of an Hyperbolic algebra ([16]).Inthis case apears infinite numbers of real roots whose dimension is more than one thousand.

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GEOMETRY OF HIGHER ORDER SPRAYS

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Abstract

In a serie of papers, [2], [3], [4], R. Miron and Gh. Atanasiu have developed a geometrization of higher order Lagrangians which parallels that of first order Lagrangians from a monograph by R. Miron and M. Anastasiei [1]. The above authors confined themselves to the time-dependent case. In this paper a short introduction to the time-dependent case is provided. First, the notion of time-dependent k -spray is introduced and characterized. Then it is shown that any time-dependent k -spray induces a nonlinear connection and any time-dependent Lagrangian of order k determines a k -spray via the Euler-Lagrange equations associated to it. An inverse problem, on the line proposed by R.M. Santilli [5], is formulated.

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1. Introduction

Let M be a smooth i.e. C^∞ real manifold of dimension n . A spray on M is a vector field on the slit tangent manifold $TM \setminus 0$, defining a second order differential equation which is homogeneous in a certain sense. When the homogeneity condition is rejected one speaks about semisprays which in turn can be defined on the whole TM . Such vector fields correspond to linear resp. nonlinear connections on M . Thus they are strongly involved in the geometry of M endowed with a connection or with a structure which induces a connection (Riemannian, Finslerian, Lagrangian and etc.). For an extensive discussion on this correspondence we refer to Ch.7 of [1].

In the last four years, R.Miron and Gh.Atanasiu developed a geometrization of higher order time-independent Lagrangians ([2],[3],[4]) which parallels the geometrization of first order Lagrangians expounded in a monograph by R.Miron and M.Anastasiu [1].

In their geometrization higher order differential equations defined by vector fields generalizing semisprays do appear. These special vector fields were called k -sprays.

In this paper k -sprays in which the time explicitly does appear are considered. Several characterizations of them are provided in §2. Then we show, in §3, that any time-dependent k -spray induces a nonlinear connection on $\mathbb{R} \times \text{Osc}^k M$. In §4 it is shown that a time-dependent Lagrangian of order k determines a time-dependent k -spray. In the end an inverse problem in the spirit of those considered by R.M.Santilli [5] is formulated.

2. Time-dependent k -spray.

Let $J_0^k(\mathbb{R}, M)$ be the manifold of k -jets over M and $\pi^k : J_0^k(\mathbb{R}, M) \rightarrow M$ the natural projection. Then $(J_0^k(\mathbb{R}, M), \pi^k, M)$ is a differentiable fibre bundle.

In a more geometric viewpoint, [2], the k -jet $j_0^k c$ with $c : I \subseteq \mathbb{R} \rightarrow M$, $0 \in I$, given in a local chart (U, x^i) , as $x^i = x^i(t)$, $t \in I$, is identified to the polynomial $x^i(0) + \frac{t}{1!} \frac{dx^i}{dt}(0) + \cdots + \frac{t^k}{k!} \frac{d^k x^i}{dt^k}(0)$. Here and in the sequel k is a fixed natural number ($k \geq 2$) and the indices $i, j, \dots$ will range from 1 to $n = \dim M$. The summation over repeated indices will be implied.

The above identification leads to the introduction of the coordinates

the manifold $J_0^k(\mathbb{R}, M)$ called also the k -osculator manifold and denoted by $\text{Osc}^k M$.

A change of coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i}) \rightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, \dots, \tilde{y}^{(k)i})$ on $\text{Osc}^k M$ is of the form:

$$(2.1) \quad \begin{array}{c} \hline k\tilde{y}^{(k)i} = (\partial_j \tilde{y}^{(k-1)i})y^{(1)j} + 2(\partial_{(1)j} \tilde{y}^{(k-1)i})y^{(2)j} + \dots + \\ \qquad \qquad \qquad + k(\partial_{(k-1)j} \tilde{y}^{(k-1)i})y^{(k)j}, \end{array}$$

where $\partial_j := \frac{\partial}{\partial x^j}$, $\partial_{(1)j} := \frac{\partial}{\partial y_{(1)j}}$, ..., $\partial_{(k)j} := \frac{\partial}{\partial y_{(k)j}}$.

The explicit appearance of time is modeled by considering the manifold $E = \mathbb{R} \times \text{Osc}^k M$ projected over $\mathbb{R} \times M$ by $\pi(t, u) = (t, x)$, $x = \pi^k(u)$, $u \in \text{Osc}^k M$. The local coordinates on E are those on $\text{Osc}^k M$ together with a new one $t \in \mathbb{R}$ subject to the condition $\tilde{t} = t$ under (2.1) with the meaning of absolute time.

Let $\pi_h^k : \mathbb{R} \times \text{Osc}^k M \rightarrow \mathbb{R} \times \text{Osc}^h M$, $h < k$, $h, k \in \mathbb{N}$, be given by $(t, x, y^{(1)}, \dots, y^{(k)}) \rightarrow (t, x, y^{(1)}, \dots, y^{(h)})$, $\pi_0^k := \pi$ and $V_1 = \ker d\pi$, $V_2 = \ker d\pi_1^k, \dots, V_k = \ker d\pi_{k-1}^k$, where $d\pi_h^k$ means the differential (tangent map) of the mapping π_h^k .

Proposition 2.1.

- $V_1 \supset V_2 \supset \cdots \supset V_k$;
- $\dim V_\beta - \dim V_{\beta-1} = n, \beta = 2, \dots, k$;
- the distribution $V_\alpha : u \rightarrow (V_\alpha)_u, u \in E, \alpha = 1, \dots, k$ is integrable.

Proof. a) follows by differentiating the equalities $\pi_{h-2}^{h-1} \circ \pi_{h-1}^h = \pi_{h-2}^h$, $h = 2, \dots, k$.

b) and c) easily result noticing that the distribution V_α is locally spanned by $(\partial_{(\alpha)i}, \partial_{(\alpha+1)i}, \dots, \partial_{(k)i})$, $\alpha = 1, \dots, k$.

Using (2.3) one checks that the following vector fields (v.f.):

$$\Gamma^{(1)} = y^{(1)i} \partial_{(k)i}, \Gamma^{(2)} = y^{(1)i} \partial_{(k-1)i} + 2y^{(2)i} \partial_{(k)i}, \Gamma^{(3)} = y^{(1)i} \partial_{(k-2)i} + 2y^{(2)i} \partial_{(k-1)i} +$$

$+3y^{(3)i}\partial_{(k)i}, \dots, \overset{(k)}{\Gamma} = y^{(1)i}\partial_{(1)i} + 2y^{(2)i}\partial_{(2)i} + \dots + ky^{(k)i}\partial_{(k)i}$, are globally defined. These are called the Liouville vector fields on E . We have also $\partial_t := \frac{\partial}{\partial t}$ as a special v.f. on E .

Now, consider the linear operators $J, \tilde{J} : T_u E \rightarrow T_u E$ defined with respect to the natural basis as follows:

$$(2.2) \quad \begin{aligned} J(\partial_t) &= 0, J(\partial_i) = \partial_{(1)i}, \dots, J(\partial_{(k-1)i}) = \partial_{(k)i}, J(\partial_{(k)i}) = 0 \\ \tilde{J}(\partial_t) &= -\overset{(k)}{\Gamma}, \tilde{J}(\partial_i) = \partial_{(1)i}, \dots, \tilde{J}(\partial_{(k-1)i}) = \partial_{(k)i}, \tilde{J}(\partial_{(k)i}) = 0. \end{aligned}$$

A direct calculation gives

Proposition 2.2.

- a) $J(\overset{(k)}{\Gamma}) = \overset{(k-1)}{\Gamma}, \dots, J(\overset{(2)}{\Gamma}) = \overset{(1)}{\Gamma}, J(\overset{(1)}{\Gamma}) = 0.$
- b) $\underbrace{J \circ J \circ \dots \circ J}_{k \text{ times}} = 0.$ The same holds for $\tilde{J}.$
- c) J is an integrable k -tangent structure.

We notice that $\tilde{J}$ is not integrable as k -tangent structure.

A time-dependent v.f. on $\text{Osc}^k M$ is a smooth mapping $X^\circ : \mathbb{R} \times \text{Osc}^k M \rightarrow T(\text{Osc}^k M)$, $(t, u) \rightarrow X^\circ(t, u) \in T_u(\text{Osc}^k M)$, $u \in \text{Osc}^k M$. It induces a v.f. on $\mathbb{R} \times \text{Osc}^k M$ by setting $X(t, u) = \partial_t + X^\circ(t, u)$.

Definition 2.1. A time-dependent k -spray is a vector field $S = \partial_t + \overset{\circ}{S}$, where $\overset{\circ}{S}$ is a time-dependent vector field on $\text{Osc}^k M$ verifying $J \overset{\circ}{S} = \overset{(k)}{\Gamma}$.

It is not difficult to see that $J \overset{\circ}{S} = \overset{(k)}{\Gamma}$ implies $\overset{\circ}{S} = y^{(1)i}\partial_i + 2y^{(2)i}\partial_{(1)i} + \dots + ky^{(k)i}\partial_{(k-1)i} - (k+1)G^i(t, x, y^{(1)}, \dots, y^{(k)})\partial_{(k)i}$, where the form of the last term was chosen for the sake of convenience. Thus we get

Proposition 2.3. A time-dependent k -spray is of the form

$$(2.3) \quad \begin{aligned} S &= \partial_t + y^{(1)i}\partial_i + 2y^{(2)i}\partial_{(1)i} + \dots + ky^{(k)i}\partial_{(k-1)i} - \\ &\quad - (k+1)G^i(t, x, y^{(1)i}, \dots, y^{(k)i})\partial_{(k)i}. \end{aligned}$$

Proposition 2.4. *A v.f. S on $E = \mathbb{R} \times \text{Osc}^k M$ is a time-dependent k -spray if and only if $JS = \overset{(k)}{\Gamma}, \hat{J}(S) = 0$.*

Proof. In a sense the implication is obvious. The converse follows noticing that any v.f. on E has the form $S = a\partial_t + \overset{\circ}{S}$, $a \in \mathbb{R}$ and $\overset{\circ}{S}$ is a time-dependent v.f. on $\text{Osc}^k M$.

Let $\Psi^{(1)i} = dx^i - y^{(1)i}dt$, $\Psi^{(2)i} = dy^{(1)i} - 2y^{(2)i}dt$, ..., $\Psi^{(k)i} = dy^{(k-1)i} - ky^{(k)i}dt$ be 1-forms on E .

The Propositions 2.3 and 2.4 yield

Proposition 2.5. *A v.f. S on $\mathbb{R} \times \text{Osc}^k M$ is a time-dependent k -spray if and only if*

$$(2.4) \quad dt(S) = 1, \Psi^{(1)i}(S) = 0, \dots, \Psi^{(k)i}(S) = 0.$$

Let $c: t \rightarrow x^i(t)$ be a curve on M and $\tilde{c}(t) = \left(t, x^i(t), \frac{dx^i}{dt}, \frac{1}{2} \frac{d^2 x^i}{dt^2}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k}\right)$

its prolongation to $\mathbb{R} \times \text{Osc}^k M$. We have

Proposition 2.6. *The curve $\tilde{c}$ is an integral curve of a time-dependent k -spray S i.e. $\frac{d\tilde{c}}{dt} = S(\tilde{c})$ if and only if the functions $t \rightarrow x^i(t)$ are solutions of the system of differential equations*

$$\frac{1}{(k+1)!} \frac{d^{k+1} x^i}{dt^{k+1}} + G^i \left(t, \frac{dx^i}{dt}, \frac{1}{2!} \frac{d^2 x^i}{dt^2}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k} \right) = 0.$$

By (2.3) the functions (G^i) uniquely determine the k -spray S . They obey the following transformation rule under (2.1):

$$(2.6) \quad (k+1)\tilde{G}^i = (k+1)(\partial_j \tilde{x}^i)G^j - (y^{(1)j}\partial_j \tilde{y}^{(k)i} + 2y^{(2)j}\partial_{(1)j} \tilde{y}^{(k)i} + \dots + ky^{(k)j}\partial_{(k-1)j} \tilde{y}^{(k)i}).$$

Conversely, a set of functions obeying (2.6) defines a k -spray.

3. Nonlinear connections and k -sprays

It is known that a time-dependent k -spray induces a nonlinear connection, [2]. It is our aim to show that this also happens for time-dependent k -sprays.

In this framework, by a nonlinear connection is meant a distribution $u \rightarrow H_u E$ on $E = \mathbb{R} \times \text{Osc}^k M$, supplementary to the vertical distribution $u \rightarrow V_u E = \ker d\pi$, that is $T_u E = H_u E \oplus V_u E$, $u \in E$.

Let v be the vertical projector on $T_u E$ and $\ell_h : \mathbb{R} \times T_{\pi(u)} M \rightarrow T_u E$ the horizontal lift ($v \circ \ell_h = 0, d\pi \circ \ell_h = \text{identity}$). Setting $\delta_i = \ell_h(\partial_i), \delta_t = \ell_h(\partial_t)$ it follows that $\delta_i - \partial_i$ and $\delta_t - \partial_t$ are vertical vector fields. Thus the local basis (δ_t, δ_i) of the horizontal distribution can be put in the form

$$(3.1) \quad \begin{aligned} \delta_i &= \partial_i - N_{(1)0}^j \partial_{(1)j} - N_{(2)0}^j \partial_{(2)j} - \cdots - N_{(k)0}^j \partial_{(k)j}, \\ \delta_i &= \partial_i - N_{(1)i}^j \partial_{(1)j} - N_{(2)i}^j \partial_{(2)j} - \cdots - N_{(k)i}^j \partial_{(k)j}, \end{aligned}$$

where the signs "–" were chosen for convenience.

The functions (N) in (3.1) obey the laws of transformation implied by

$$(3.2) \quad \delta_i = (\partial_i \tilde{x}^j) \tilde{\delta}_j, \quad \hat{\delta}_t = \delta_t.$$

We omit them since we shall use a nonlinear connection as given by its dual coefficients (M) to be introduced below.

Let be $N_0 = \text{span}(\delta_i)$, $N_1 = J(N_0), \dots, N_{k-1} = J^{k-1}(N_0)$ and let us put $\delta_{(1)i} = J(\delta_i)$, $\delta_{(2)i} = J^2(\delta_i), \dots, \delta_{(k-1)i} = J^{k-1}(\delta_i)$. We have the decomposition

$$(3.3) \quad T_u E = \text{span}(\delta_t) \oplus N_0 \oplus N_1 \oplus \cdots \oplus N_{k-1} \oplus V_k,$$

and the local vector fields $(\delta_t, \delta_i, \delta_{(1)i}, \dots, \delta_{(k-1)i}, \delta_{(k)i})$ provide a basis of $T_u E$, adapted to this decomposition.

The dual of this basis is $(dt, dx^i, \delta y^{(1)i}, \dots, \delta y^{(k)i})$ with

$$\begin{aligned}
\delta y^{(1)i} &= dy^{(1)i} + M_{(1)j}^i dx^j + M_{(1)0}^i dt, \\
\delta y^{(2)i} &= dy^{(2)i} + M_{(1)j}^i dy^{(1)j} + M_{(2)j}^i dx^j + M_{(2)0}^i dt, \\
\delta y^{(k)i} &= dy^{(k)i} + M_{(1)j}^i dy^{(k-1)j} + M_{(2)j}^i dy^{(k-2)j} + \dots + \\
&\quad + M_{(k)j}^i dx^j + M_{(k)0}^i dt,
\end{aligned}
\tag{3.4}$$

The functions $(M_{(\alpha)})$, $\alpha = 1, \dots, k$, in (3.4) are called the dual coefficients of the given nonlinear connection.

The duality of the above basis yields

$$\begin{aligned}
 (3.5) \quad & M_{(1)i}^i = N_{(1)j}^i, \\
 & M_{(2)j}^i = N_{(2)j}^i + N_{(1)s}^i M_{(1)j}^s, \\
 & \vdots \\
 & M_{(k)j}^i = N_{(k)j}^i + N_{(k-1)s}^i M_{(1)j}^s + \cdots + N_{(2)s}^i M_{(k-2)j}^s + N_{(1)s}^i M_{(k-1)j}^s, \\
 & M_{(1)0}^i = N_{(1)0}^i, \quad M_{(2)0}^i = N_{(2)0}^i + N_{(1)0}^s M_{(1)s}^i, \dots \\
 & M_{(k)0}^i = N_{(k)0}^i + N_{(k-1)0}^s M_{(1)s}^i + \cdots + N_{(2)0}^s M_{(k-2)s}^i + N_{(1)0}^s M_{(k-1)s}^i.
 \end{aligned}$$

These formulae can be reversed in order to express the functions (N_α) with the help of the functions (M_α) . In many cases a nonlinear connection appears as given by the dual coefficients (M_α) . The laws of transformation of the coefficients (M_α) under (2.1) are implied by

$$(3.6) \quad dt = d\tilde{t}, d\tilde{x}^i = (\partial_j \tilde{x}^i) dx^j, \delta \tilde{y}^{(\alpha)i} = (\partial_j \tilde{x}^i) \delta y^{(\alpha)j}, \alpha = 1, \dots, k.$$

These are as follows:

$$\begin{aligned}
 (3.7) \quad & M_{(1)j}^s (\partial_s \tilde{x}^j) = \tilde{M}_{(1)s}^i (\partial_j \tilde{x}^s) + \partial_j \tilde{y}^{(1)i}, \\
 & M_{(2)j}^s (\partial_s \tilde{x}^j) = \tilde{M}_{(2)s}^i (\partial_j \tilde{x}^s) + \tilde{M}_{(1)s}^i (\partial_j \tilde{y}^{(1)s}) + \partial_j \tilde{y}^{(2)i}, \\
 & \dots \dots \dots \\
 & \tilde{M}_{(\alpha)0}^i = (\partial_j \tilde{x}^s) \tilde{M}_{(\alpha)0}^j, \quad \alpha = 1, \dots, k.
 \end{aligned}$$

It follows that for any fixed t the coefficients $(M_{(\alpha)j}^i)$, $\alpha = 1, \dots, k$, define a nonlinear connection on $\text{Osc}^k M$, while $(M_{(\alpha)0}^i)$, appear as components of a vector field on $\text{Osc}^k M$. Thus for any fixed t the v.f. $\overset{\circ}{S}$, which is nothing but a k -spray on $\text{Osc}^k M$, determines the coefficients $(M_{(\alpha)j}^i)$, $\alpha = 1, \dots, k$, according to the following formulae, established by R. Miron and Gh. Atanasiu [2]:

$$\begin{aligned}
 (3.8) \quad & M_{(1)j}^i = \partial_{(k)j} G^i, \\
 & M_{(2)j}^i = \frac{1}{2} (\overset{\circ}{S} M_{(1)j}^i + M_{(1)s}^i M_{(1)j}^s), \\
 & \vdots \\
 & M_{(k)j}^i = \frac{1}{k} (S M_{(k-1)j}^i + M_{(1)s}^i M_{(k-1)j}^s).
 \end{aligned}$$

Now, by (2.6) it follows that

$$(3.9) \quad M_{(1)0}^i = \partial_t G^i, M_{(2)0}^i = \partial_t \circ \partial_t G^i, \dots, M_{(k)0}^i = \underbrace{\partial_t \circ \cdots \circ \partial_t}_{k \text{ times}} G^i$$

transform under (2.1) as the components of a covector so these functions can be taken as coefficients for defining a nonlinear connection. Thus we get

Theorem 3.1. *A time-dependent k -spray $S = \partial_t + \overset{\circ}{S}$ induces a nonlinear connection on E whose dual coefficients $(M_{(\alpha)j}^i)$ and $(M_{(\alpha)0}^i)$, $\alpha = 1, \dots, k$, are given by (3.8) and (3.9).*

Finally we notice the possibility to consider a connection similar to the Berwald connection from Finsler geometry. This is a linear connection D in the vertical bundle over E defined with respect to the adapted basis as follows:

$$(3.10) \quad \begin{aligned} D_{\delta_i} \delta_{(\alpha)i} &= L_{i0}^h \delta_{(\alpha)h}, L_{i0}^h = \partial_{(k)i} N_0^h, \\ D_{\delta_j} \delta_{(\alpha)i} &= L_{ij}^h \delta_{(\alpha)h}, L_{ij}^h = \delta_{(1)j} N_{(1)i}^h, \\ D_{\delta_{(\beta)j}} \delta_{(\alpha)i} &= 0. \end{aligned}$$

This connection can be viewed as defined by a k -spray via the equations

$$(3.11) \quad \begin{aligned} N_{(1)0}^h &= M_{(1)0}^h = \partial_i G^h, \\ N_{(1)i}^h &= M_{(1)i}^h = \partial_{(k)i} G^h \end{aligned}$$

It can be seen that this connection is flat when there exist coordinates on M in which $G^i(t, x, y^{(1)}, \dots, y^{(k)}) = A_j^i(t) y^{(k)j} + \overset{\circ}{G}(t, x, y^{(1)}, \dots, y^{(k-1)})$.

4. Lagrangians of order k and k -sprays

We show that a time-dependent Lagrangian of order k induces a time-dependent k -spray. The case time-dependent was solved by R.Miron and G.Atanasiu [3].

A time-dependent Lagrangian is a smooth function $L: \mathbb{R} \times \text{Osc}^k M \rightarrow \mathbb{R}$, $(t, x, y^{(1)}, \dots, y^{(k)}) \rightarrow L(t, x, y^{(1)}, \dots, y^{(k)})$ with the property that the matrix $(g_{ij}(t, x, y^{(1)}, \dots, y^{(k)})) := (\frac{1}{2} \partial_{(k)i} \partial_{(k)j} L)$ has rank n . The solutions of the variational problem $\delta \int_{t_0}^{t_1} L dt = 0$ are given by the well-known Euler-Lagrange equations

$$(4.1) \quad \begin{aligned} \overset{\circ}{E}_i(L) &:= \partial_i L - \frac{d}{dt}(\partial_{(1)i} L) + \frac{d^2}{dt^2}(\partial_{(2)i} L) - \dots + (-1)^k \frac{d^k}{dt^k}(\partial_{(k)i} L) = 0, \\ y^{(1)i} &= \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{d^k x^i}{dt^k}. \end{aligned}$$

Besides $\overset{\circ}{E}_i$, viewed as an operator on L , one considers the following operators

$$(4.2) \quad \begin{aligned} \overset{1}{E}_i &= \sum_{\alpha=1}^k (-1)^\alpha \frac{1}{\alpha!} \binom{\alpha}{\alpha-1} \frac{d^\alpha}{dt^\alpha} \partial_{(\alpha)i}, \\ \overset{2}{E}_i &= \sum_{\alpha=2}^k (-1)^\alpha \frac{1}{\alpha!} \binom{\alpha}{\alpha-2} \frac{d^{\alpha-2}}{dt^{\alpha-2}} \partial_{(\alpha)i}, \\ \dots\dots\dots \\ \overset{k-1}{E}_i &= (-1)^{k-1} \frac{1}{(k-1)!} \left(\partial_{(k-1)i} - \frac{d}{dt} \partial_{(k)i} \right), \\ \overset{k}{E}_i &= (-1)^k \frac{1}{k!} \partial_{(k)i}. \end{aligned}$$

These operators were introduced in [3]. Each one applied to L and computed along of the curve $\sigma^* = \left(t, x^i(t), \frac{1}{1!} \frac{dx^i}{dt}, \frac{1}{2!} \frac{d^2 x^i}{dt^2}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k} \right)$ which extends to E a solution $t \rightarrow \sigma(t) = (t, x^i(t))$ of (4.1), provide a d -covector field. All these are called the Craig-Synge covectors.

Proposition 4.1. *Along of the curve σ^* , the d -covector $\overset{k-1}{E}_i(L)$ takes the form*

$$(4.3) \quad \overset{k-1}{E}_i(L) = (-1)^{k-1} \frac{1}{(k-1)!} \left(\partial_{(k-1)i} L - \tilde{\Gamma}(\partial_{(k)i} L) - \frac{2}{k!} g_{ij} \frac{d^{k+1} x^i}{dt^{k+1}} \right),$$

where $\tilde{\Gamma} = \partial_t + \Gamma$, $\Gamma = y^{(1)i} \partial_i + 2y^{(2)i} \partial_{(1)i} + \dots + ky^{(k)i} \partial_{(k)i}$.

Proof. By expanding the derivative with respect to t in $\overset{k-1}{E}_i$ from (4.2) one gets the form (4.3).

Proposition 4.2. *A time-dependent Lagrangian determines a k -spray.*

Proof. It easily results that the equations $g^{ij} \overset{k-1}{E}_j(L) = 0$ are equivalent to

$$\frac{2}{k!} \frac{d^{k+1} x^i}{dt^{k+1}} + g^{ij} (\tilde{\Gamma}(\partial_{(k)j} L) - \partial_{(k-1)j}) = 0.$$

Comparing these equations with (2.5) it comes out that we can take

$$G^i = \frac{1}{2(k+1)} g^{ij} (\hat{\Gamma}(\partial_{(k)j} L) - \partial_{(k-1)j} L)$$

and by a direct calculation one checks that these (G^i) satisfy (2.6) under (2.1), q.e.d.

The Theorem 3.1 and Proposition 4.2 yield

Corollary 4.1. *Every time-dependent Lagrangian determines a non-linear connection.*

The following problem naturally arises. Given a k -spray S in which conditions there exists a time-dependent Lagrangian of order k inducing just the k -spray S ? The problem is clearly connected with the *inverse problems* treated by R.M. Santilli [5].

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**FRACTIONAL CALCULUS AND ITS APPLICATIONS
TO DIFFERENTIAL EQUATIONS**

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Abstract

Fractional calculus deals with the integration and differentiation of arbitrary order. We use the concept of fractional calculus to solve a number of ordinary and partial differential equations. Various special differential equations considered recently by K. Nishimoto [Fractional Calculus, Vol. 1, (1984), Vol. 2(1987), Vol. 3(1989) and Vol. 4(1991), Descartes Press, Koriyama, Japan] and J. Duran & Kalla, and others in the Journal of Fractional Calculus, follow as particular cases of our results.

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1. Introduction

The concept of fractional calculus can be traced back to L' Hopital's curiosity about the meaning of Leibniz's n th derivative notation $\frac{d^n y}{dx^n}$ when $n = \frac{1}{2}$. Leibniz, Euler, Laplace, Lacroix, and Fourier made mention of derivatives of arbitrary order, but the first use of fractional operators was made by Niels Abel [6]. Abel applied the fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem (i.e. the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of the starting point).

The fractional calculus deals with the derivatives and integrals of arbitrary orders, called differintegrals. Thus the differintegral of order α (real or complex) is a generalization of the ordinary n -th derivative and n -times integral. There are several definitions of an integral of arbitrary order, one of the simplest is the Riemann-Liouville operators [4,6,18]; let $f(x) \in L_1(a, b)$

$$(R_{a+}^{\alpha} f)(x) = (I_{a+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a \quad (1.1)$$

$$(R_{a-}^{\alpha} f)(x) = (I_{a-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < a \quad (1.2)$$

where $\alpha > 0$, are called fractional integrals of order α . Sometimes they are called as left-sided and right-sided fractional integrals respectively.

Besides the above Riemann-Liouville definition of fractional operators, many other modifications and generalizations can be found in the literature. The most useful classical fractional integrals are due to Erdelyi-kober [4,5,17],

$$I_{\eta,\alpha}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t) dt \quad (1.3)$$

where $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\eta) > -\frac{1}{2}$. Several authors, like Love, Kalla & Saxena, Saigo, McBride etc. [4,5,6,16,17,18] have studied and used different forms of the so called hypergeometric operators of fractional integration;

$$If(x) = \frac{ux^{-\gamma-1}}{\Gamma(1-\delta)} \int_0^x {}_2F_1\left(\delta, \beta+m; \eta, a\left(\frac{t}{x}\right)^u\right) t^\gamma f(t) dt \quad (1.4)$$

involving the Gauss hypergeometric function.

The five volume work published by Nishimoto [7] contains an interesting account of the theory and applications of fractional calculus in mathematical analysis, specially in differential equations and the summation of series. He defines the differintegral in the following form:

$$f_\alpha = {}_c f_\alpha(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_c \frac{f(\xi)}{(\xi-z)^{\alpha+1}} d\xi \quad (1.5)$$

where $f(z)$ is a regular function and it has no branch points inside and on $C(C = \{C-, C+\}, C-$ is an integral curve along the cut joining two points z and $-\infty + i \operatorname{Im}(z)$ and $C+$ is an integral curve along the cut joining two points z and

$\infty + i \operatorname{Im}(z)$), where $\alpha \notin Z^-, \alpha \in R$ and

$$f_{-n} = \lim_{\alpha \rightarrow -n} f_\alpha \quad (n \in Z^+)$$

(Z^- and Z^+ are the set of negative and positive integers respectively) $\xi \neq z$,

$$-\pi \leq \arg(\xi - z) \leq \pi \quad \text{for } C_-$$

and

$$0 \leq \arg(\xi - z) \leq 2\pi \quad \text{for } C_+.$$

$f_\alpha(\alpha > 0)$ is the fractional derivative of order α , and $f_\alpha(\alpha < 0)$ is the fractional integral of order α , if f_α exists, the principal value of f will be considered for many valued functions.

Nishimoto and his associates have obtained the solution of several special differential equations by invoking the fractional calculus. Kalla & Guerra [3] have obtained the solution of a class of third order differential equation by transforming to its operational form and using the Riemann-Liouville operator. Results given earlier by Al-Bassam and Nishimoto & Kalla [12] follow as special cases of [3]. Recently some authors [2,7,19] have treated special linear ordinary differential equations of n-th order and obtained their solution by an appeal to the differintegral formula for the product of two functions [7]:

Let $u(z)$ and $v(z)$ be analytic and are single valued functions respectively, then $\{uv\}_\alpha$ is fractional derivative of product uv for $\text{Re}(\alpha) > 0$, original product uv for $\alpha = 0$ and fractional integral of the product uv for $\text{Re}(\alpha) < 0$. If u_α and v_α exist, $\{uv\}_\alpha$ is given by:

$$\{uv\}_\alpha = \{(uv)_\alpha, (vu)_\alpha\}$$

$$\left\{ \begin{array}{l} (uv)_\alpha = \sum_{n=0}^{\infty} p(\alpha, n) u_{\alpha-n} v_n \\ (vu)_\alpha = \sum_{n=0}^{\infty} p(\alpha, n) v_{\alpha-n} u_n \\ \text{and } p(\alpha, n) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \\ z, \alpha \in C. \end{array} \right. \quad (1.7)$$

Our aim here is to obtain particular solution of some general class of differential equations. In section 2 we consider differential equations of order two and three. The section 3 deals with a general class of differintegral equations. The results of this section are used to obtain solution of some partial differential equations in the section 4. Some particular cases are cited. The solutions are obtained using the definition (1.5) and the product rule (1.7). For the sake of brevity, we state only the solutions without details in sections 3 and 4. The details of proof and more special cases will be given elsewhere.

2. Second and third order differential equations

We obtain particular solutions of some general class of second and third order linear differential equations.

Theorem 2.1. If $f_\alpha (\neq 0)$ exists, then the nonhomogeneous second order linear ordinary differential equation,

$$\begin{aligned} & \varphi_2(a_1 z^3 - a_2 z^2 + a_3 z + a_4) + \\ & \varphi_1\{2a_1(1-\alpha)z^2 + \{a_2(\alpha-1) + (1+\beta+\gamma)\}z - \alpha\beta\gamma\} \\ & + \varphi\{a_1\alpha(\alpha-1)z - \alpha(1+\beta+\gamma)\} = f \end{aligned} \quad (2.1)$$

has a particular solution of the form

$$\varphi = \left(\left(f_\alpha \cdot \frac{1}{a_1 z^3 - a_2 z^2 + a_3 z + a_4} e^{H(z)} \right)_{-1} \cdot e^{-H(z)} \right)_{-1-\alpha} \quad (2.2)$$

$$\text{where } H(z) = \int \frac{a_1(\alpha+2)z^2 + ((1+\beta+\gamma) - a_2(\alpha+1))z - \alpha(\beta\gamma - a_3)}{a_1 z^3 - a_2 z^2 + a_3 z + a_4} dz \quad (2.3)$$

$\varphi = \varphi(z)$, $f = f(z)$, $z \in \mathbb{C}$, $a_1, a_2, a_3, a_4, \alpha, \beta$ and γ are arbitrary constants.

Proof: Let $\varphi = w_{-\alpha} = w_{-\alpha}(z)$

(2.4)

then $\varphi_1 = w_{1-\alpha}$

(2.5)

and $\varphi_2 = w_{2-\alpha}$

(2.6)

Using (1.7), (2.4), (2.5), and (2.6); (2.1) can be expressed as,

$$\begin{aligned} & (w_2 \cdot a_1 z^3)_{-\alpha} - (w_2 \cdot a_2 z^2)_{-\alpha} + (w_2 \cdot a_3 z)_{-\alpha} + a_4 (w_2)_{-\alpha} \\ & + (w_1 \cdot z^2)_{-\alpha} (a_1(\alpha+2)) - (w_1 \cdot z)_{-\alpha} \{a_2(\alpha+1) - (1+\beta+\gamma)\} \\ & - (w_1)_{-\alpha} (\alpha\beta\gamma - a_3\alpha) = f. \end{aligned} \quad (2.7)$$

Hence,

$$w_2 + w_1 \cdot \frac{a_1(\alpha+2)z^2 + ((1+\beta+\gamma) - a_2(\alpha+1))z - \alpha(\beta\gamma - a_3)}{a_1z^3 - a_2z^2 + a_3z + a_4}$$

$$= f_\alpha \frac{1}{a_1z^3 - a_2z^2 + a_3z + a_4} \quad (2.8)$$

$$\text{since } (w_m \cdot z^n)_{-\alpha} = \sum_{k=0}^n \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-k)\Gamma(k+1)} (w_m)_{-\alpha-k} (z^n)_k \quad (2.9)$$

where $n \in \mathbb{Z}^+ \cup \{0\}$.

Setting $w_1 = u$, we get $w_2 = u_1$,
(2.10)

substituting (2.10) into (2.9) we obtain

$$u_1 + u \cdot \frac{a_1(\alpha+2)z^2 + ((1+\beta+\gamma) - a_2(\alpha+1))z - \alpha(\beta\gamma - a_3)}{a_1z^3 - a_2z^2 + a_3z + a_4}$$

$$= f_\alpha \frac{1}{a_1z^3 - a_2z^2 + a_3z + a_4} \quad (2.11)$$

A particular solution of (2.11) is given by

$$u = \left(f_\alpha \frac{1}{a_1z^3 - a_2z^2 + a_3z + a_4} e^{H(z)} \right)_{-1} \cdot e^{-H(z)} \quad (2.12)$$

where $H(z)$ is given by (2.3).

Hence, from (2.4), (2.10) and (2.12), we obtain $\varphi(z)$ as in (2.2).

Following the same procedure we can easily derive the following result for the corresponding homogeneous case.

Theorem 2.2. The homogeneous differential equation

$$\begin{aligned} & \varphi_2(a_1 z^3 - a_2 z^2 + a_3 z + a_4) \\ & + \varphi_1 \left\{ 2a_1(1-\alpha)z^2 + (a_2(\alpha-1) + (1+\beta+\gamma))z - \alpha\beta\gamma \right\} \\ & + \varphi(a_1\alpha(\alpha-1)z - \alpha(1+\beta+\gamma)) = 0 \end{aligned} \quad (2.13)$$

has a solution of the form

$$\varphi = \left(e^{-H(z)} \right)_{-\alpha-1} \quad (2.14)$$

where $H(z)$ is a function given by (2.3),

$\varphi = \varphi(z)$, $z \in \mathbb{C}$, $a_1, a_2, a_3, a_4, \alpha, \beta$ and γ are arbitrary constants.

Theorem 2.3. If $f_{-\alpha} (\neq 0)$ exists, then the nonhomogeneous third order linear differential equation

$$\begin{aligned} & \varphi_3(a_1 z^3 + a_2 z^2 + a_3 z + a_4) \\ & + \varphi_2 \left\{ (\alpha(2a_1 + 1) + \beta + \gamma + 2(a_1 - 1))z^2 + \{a_2(\alpha + 1) + \beta\gamma\}z + \alpha a_3 \right\} \\ & + \varphi_1 \left\{ (\alpha(\alpha(a_1 + 2) + a_1 - 4 + 2(\beta + \gamma)))z + \alpha\beta\gamma \right\} \\ & + \varphi \{ \alpha(\alpha - 1)(\alpha + \beta + \gamma - 2) \} = f \end{aligned} \quad (2.15)$$

has a particular solution of the form

$$\varphi = \left[\left(f_{-\alpha} \frac{1}{a_1 z^3 + a_2 z^2 + a_3 z + a_4} e^{H(z)} \right)_{-1} \cdot e^{-H(z)} \right]_{\alpha-2} \quad (2.16)$$

where

$$H(z) = \int \frac{k_1 z + k_2}{a_1 z^3 + a_2 z^2 + a_3 z + a_4} dz, \quad (2.17)$$

$\varphi = \varphi(z)$, $f = f(z)$, $z \in C$, $a_1, a_2, a_3, a_4, \alpha, \gamma$ and γ are arbitrary constants,

$$k_1 = (\alpha - 2)(1 - a_1) + \beta + \gamma, \quad (2.18)$$

$$\text{and } k_2 = \beta\gamma - a_2(\alpha - 1). \quad (2.19)$$

Theorem 2.4. The homogeneous differential equation

$$\begin{aligned} \varphi_3(a_1 z^3 + a_2 z^2 + a_3 z + a_4) + \varphi_2((\alpha(2a_1 + 1) + \beta + \gamma + 2(a_1 - 1))z^2) \\ + \varphi_1((\alpha(\alpha(a_1 + 2) + a_1 - 4 + 2(\beta + \gamma))z + \alpha\beta\gamma)) \\ + \varphi(\alpha(\alpha - 1)(\alpha + \beta + \gamma - 2)) = 0 \end{aligned} \quad (2.20)$$

has a solution of the form

$$\varphi = \left(e^{H(z)} \right)_{\alpha-2} \quad (2.21)$$

where $H(z)$ is the function given by (2.17), $\varphi = \varphi(z)$, $z \in C$, $a_1, a_2, a_3, a_4, \alpha, \beta$ and γ are arbitrary constants.

3. Differential equations of order n.

Theorem 3.1. If $f \in F$ and $f_{-(\alpha+\beta)} (\neq 0)$ exists, then the nonhomogeneous differintegral equation

$$\begin{aligned} \varphi_m(a_0 z^n + \dots + a_n) + \sum_{k=1}^n \varphi_{m-k} \left\{ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\ \left. + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+2-k)\Gamma(k)} (b_0 z^n + \dots + b_n)_{k-1} \right\} + b_0 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-n)} \varphi_{m-(n+1)} = f \end{aligned} \quad (3.1)$$

has a particular solution of the form

$$\varphi = \left(\left(f_{-(\alpha+\beta)}(a_0 z^n + \dots + a_n)^{-1} \cdot e^{H(z)} \right)_{-1} \cdot e^{-H(z)} \right)_{\alpha+\beta+1-m} \quad (3.2)$$

where
$$H(z) = \int \frac{b_0 z^n + \dots + b_n}{a_0 z^n + \dots + a_n} dz \quad (3.3)$$

$\varphi = \varphi(z), f = f(z), z \in C, a_i, b_i (0 \leq i \leq n), \alpha, \beta$ are arbitrary constants, $m \in Z, n \in Z^+$.

Theorem 3.2. The homogeneous differintegral equation

$$\begin{aligned} \varphi_m(a_0 z^n + \dots + a_n) + \sum_{k=1}^n \varphi_{m-k} \left\{ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\ \left. + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+2-k)\Gamma(k)} (b_0 z^n + \dots + b_n)_{k-1} \right\} + b_0 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-n)} \varphi_{m-(n+1)} = 0 \end{aligned} \quad (3.4)$$

has a solution of the form

$$\varphi = K \left(e^{-H(z)} \right)_{\alpha+\beta+1-m}$$

where $H(z)$ is the function given by (3.3), $\varphi = \varphi(z), z \in C, a_i, b_i (0 \leq i \leq n), \alpha, \beta$ are arbitrary constants, $m \in Z, n \in Z^+$ and K is an arbitrary constant.

4. Partial differential equations

In this section we use the results obtained in section 3, to prove the following theorems.

Theorem 4.1 Generalized Gauss type P.D.E.

$$\begin{aligned}
 (u)_{m(z)}(a_0 z^n + \dots + a_n) + \sum_{k=1}^{n-1} (u)_{(m-k)(z)} \left\{ \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\
 \left. + \frac{\Gamma(\rho+1)}{\Gamma(\rho+2-k)\Gamma(k)} (b_1 z^{n-1} + \dots + b_n)_{k-1} \right\} \\
 + \delta(u)_{(m-n)(z)} = \left(A(u)_{2(t)} + B(u)_{1(t)} \right)_{(m-n)(z)} \quad (4.1)
 \end{aligned}$$

has solutions of the form

$$\text{(i)} \quad u(z, t) = K \left(e^{-H(z)} \cdot e^{\left\{ -B \pm \sqrt{B^2 + 4A(\delta-C)} \right\} t / 2A} \right)_{(\rho+1-m)(z)} \quad \text{for } AB \neq 0 \quad (4.2)$$

$$\text{(ii)} \quad u(z, t) = K \left(e^{-H(z)} \cdot e^{\pm \left\{ \sqrt{(\delta-C)/A} \right\} t} \right)_{(\rho+1-m)(z)} \quad \text{for } A \neq 0 \text{ and } B = 0 \quad (4.3)$$

$$\text{(iii)} \quad u(z, t) = K \left(e^{-H(z)} \cdot e^{\left\{ (\delta-C)/B \right\} t} \right)_{(\rho+1-m)(z)} \quad \text{for } A = 0 \text{ and } B \neq 0 \quad (4.4)$$

where $u = u(z, t)$, $z \in C$, $\rho, \delta, a_0, \dots, a_n, b_1, \dots, b_n, A, B$ are given constants, K is an arbitrary constant and

$$H(z) = \int \frac{b_1 z^{n-1} + \dots + b_n}{a_0 z^n + \dots + a_n} dz, \quad C = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-n)} + \frac{\Gamma(\rho+1)}{\Gamma(\rho+2-n)} b_1 \quad (4.5)$$

Theorem 4.2 The partial differential equation

$$\begin{aligned} & (u)_{m(z)}(a_0 z^n + \dots + a_n) + \sum_{k=1}^n (u)_{(m-k)(z)} \left\{ \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\ & \left. + \frac{\Gamma(\rho+1)}{\Gamma(\rho+2-k)\Gamma(k)} (b_0 z^n + \dots + b_n)_{k-1} \right\} \\ & + \delta(u)_{(m-(n+1))(z)} = \left(A(u)_{2(t)} + B(u)_{1(t)} \right)_{(m-(n+1))(z)} \end{aligned} \quad (4.6)$$

has a solutions of the form

$$(i) \quad u(z, t) = K \left(e^{-H(z)} e^{\left\{ -B \pm \sqrt{B^2 + 4A(\delta-D)} \right\} t / 2A} \right)_{(\rho+1-m)(z)} \quad \text{for } AB \neq 0 \quad (4.7)$$

$$(ii) \quad u(z, t) = K \left(e^{-H(z)} e^{\pm \left\{ \sqrt{(\delta-D)/A} \right\} t} \right)_{(\rho+1-m)(z)} \quad \text{for } A \neq 0 \text{ and } B = 0 \quad (4.8)$$

$$(iii) \quad u(z, t) = K \left(e^{-H(z)} e^{(\delta-D)t/B} \right)_{(\rho+1-m)(z)} \quad \text{for } A = 0 \text{ and } B \neq 0 \quad (4.9)$$

where $u = u(z, t)$, $z \in C$, $\rho, \delta, a_0, \dots, a_n, b_0, \dots, b_n, A, B$ are given constants, K is an arbitrary constant and

$$H(z) = \int \frac{b_0 z^n + \dots + b_n}{a_0 z^n + \dots + a_n} dz, \quad D = b_0 \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-n)} \quad (4.10)$$

Theorem 4.3. The homogeneous partial differential equation

$$\begin{aligned} (u)_{m(z)}(a_0 z^n + \dots + a_n) + \sum_{k=1}^{n-1} (u)_{(m-k)(z)} \left\{ \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\ \left. + \frac{\Gamma(\rho+1)}{\Gamma(\rho+2-k)\Gamma(k)} (b_1 z^{n-1} + \dots + b_n)_{k-1} \right\} + \delta(u)_{(m-n)(z)} = 0 \end{aligned}$$

has a solution of the form

$$u(z, t) = K \left(e^{-H(z)} \right)_{\rho+1-m} e^{\lambda t} \quad (\lambda \neq 0)$$

where $H(z)$ is given by (4.5), $z \in C$, $a_0, \dots, a_n, b_1, \dots, b_n, \rho$ and K are arbitrary constants, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}^+$.

Theorem 4.4. The homogeneous partial differential equation

$$\begin{aligned} (u)_{m(z)}(a_0 z^n + \dots + a_n) + \sum_{k=1}^n (u)_{(m-k)(z)} \left\{ \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-k)\Gamma(k+1)} (a_0 z^n + \dots + a_n)_k \right. \\ \left. + \frac{\Gamma(\rho+1)}{\Gamma(\rho+2-k)\Gamma(k)} (b_0 z^n + \dots + b_n)_{k-1} \right\} + \delta(u)_{(m-(n+1))(z)} = 0 \end{aligned}$$

has a solution of the form

$$u(z, t) = K \left(e^{-H(z)} \right)_{\rho+1-m} e^{\lambda t} \quad (\lambda \neq 0)$$

where $H(z)$ is given by (4.10) $z \in C$, $a_0, \dots, a_n, b_0, \dots, b_n$, ρ and K are arbitrary constants, $m \in Z$, and $n \in Z^+$.

5. Special Cases

Several known results can be derived from the differential equations given in previous sections.

For example:

In Theorem 2.1 if we set $a_1 = 1$, $a_2 = 1$, $a_3 = 0$, and $a_4 = 0$ we obtain a known result recently given by Nishimoto [7].

Also setting $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, and $a_4 = b$ in Theorem 2.1 we obtain the result recently given by Al-Saqabi and Kalla [1].

In Theorem 2.3 If we set, $a_1 = 1$, $a_4 = -1$, and $a_1 = a_3 = 0$ we obtain the result given recently by Nishimoto [8]. $a_1 = 0$, $a_1 = 1$, $a_3 = -1$, and $a_4 = 0$ in Theorem 2.3 leads to the result of Nishimoto [8].

Further, If we set $a_0 = cd, a_1 = -(bc + ad), a_2 = ab$,

$a_3 = \dots = a_n = 0, b_0 = A, b_1 =$

$B, b_2 = C$, and $b_3 = \dots = b_n = 0$ in Theorem 3.1 we obtain the result given by Nishimoto [7].

Let $m = n = 2, a_0 = 1, a_1 = a_2 = 0, A = 0, \delta = 1, b_1 = -2\ell$ and $b_2 = -1$ in Theorem 4.1 we obtain the result given by Nishimoto in [7].

Also taking $a_0 = 1, a_1 = -\delta, a_2 = \dots = a_n = 0, b_0 = \beta + \gamma, b_1 = -\beta\gamma, b_2 = \lambda\beta\gamma$, and $b_3 = \dots = b_n = 0$ in Theorem 4.2 we obtain a known result recently gives by Duran and Kalla [2].

In a similar way a number of results can be derived from our general theorems by choosing suitable values for the parameters.

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EMBEDDING OF LIE ALGEBRAS IN GENERALIZED
STRUCTURABLE ALGEBRAS

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Abstract

In a preceding paper, the first named author has introduced the *generalized structural algebras*, while the second named author has introduced the *isotopies of Lie algebras*. In this paper we combine the two analyses, submit the notion of *isogeneralized structurable algebras*, and show that they include Lie algebras, all their axiom-preserving generalizations of graded, supersymmetric and isotopic type, as well as numerous other algebras.

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1.Introduction

As well known (see,e.g.,[1]), *Lie algebras* L can be conceived as a linear vector space(hereon assumed to be not-necessarily finite-dimensional) with elements $x, y, z, \dots$ and abstract product $[x, y] = xy - yx$ over a field Φ (hereon assumed of characteristic different than 2 or 3)which verifies the axioms

$$\begin{aligned}[x, y] + [y, x] &= 0, \\ [[x, y], z] + [[y, z], x] + [[z, x], y] &= 0,\end{aligned}\tag{1}$$

for all $x, y, z \in L$, where the product xy is generally assumed to be that of the enveloping associative algebra $E(L)$ of L .

A reason for the importance of Lie algebras in physics is that their product $[x, y]$ characterized the brackets of the envolution with respect to time t , i.e., $dx/dt = [x, h]$, where the element h is called the hamiltonian and generally represents the total energy.

In a preceding paper [2], N.Kamiya has introduced the *generalized structurable algebras*, defined as the generally nonassociative algebras A over Φ equipped with a nontrivial derivation $D(x, y)$ verfyng the axioms

$$\begin{aligned}D(x, y) + D(y, x) &= 0, \\ D(xy, z) + D(yz, x) + D(zx, y) &= 0\end{aligned}\tag{2}$$

for all $x, y, z \in A$.

It is evident that axioms (2) are a generalization of the conventional Lie axioms (1). As a reslut, the generalized structurable algebras are a significant generalization of Lie algebas. In particular, N.Kamiya has shown that, besides

the Lie algebras, the generalized structurable algebras include in thier classification: the *structurable algebras* due to B.N.Allison; the more general *standard algebras* due to H.Asano; the *generalized triple systems* due to K.Yamaguti; the *Malcev algebras*; the *conservative algebras* due to I.L.Kantor; the *commutative Jordan algebras*; and others.

Independently from the above study, R.Santilli [3] introduced the *isotopies of Lie(and other) algebras*, which can be defined as the lifting of the Lie algebra L with elements $x, y, z, \dots$, and product $[x, y] = xy - yx$ over Φ into infinitely possible *Lie-isotopic algebras* L which coincide with L as vector spaces, but are defined wiht respect to infinitely possible new product $[x\hat{,}y] = x * y - y * x$ such to preseve the original axioms (1) (thus illustrating the term "isotopy" from the Greek $\iota\sigma\omicron\zeta\ \tau\omicron\pi\omicron\zeta$)

$$[x\hat{,}y] + [y\hat{,}x] = 0$$

$$[[x\hat{,}y]\hat{,}z] + [[y\hat{,}z]\hat{,}x] + [[z\hat{,}x]\hat{,}y] = 0, \quad (3)$$

for all $x, y, z \in L$.

The following isotopies of the associative product xy , and therefore of the attached Lie product $xy - yx$, were identified in ref.[3]:

1) the *scalar isitopy*

$$x * y = \alpha xy, \ \alpha \in \Phi \neq 0, \quad (4)$$

2) the *non trival isotopy*

$$x * y = xTy, \quad (5)$$

where T is a fixed, nowhere null and Hermitean element generally outside L ; and

3) the *idempotent isotopy*

$$x * y = wxwyy, w^2 = w, w \neq 0, \quad (6)$$

as well as all their possible combinations, e.g., $x * y = \alpha wxwTwyw$. Note that isotopies (4) and (5) admit the left and right until $\hat{I} = \alpha^{-1}$ and $\hat{I} = T^{-1}$, respectively, called *isounit*, while no unit is admitted by isotopy (6).

R.M.Santilli [4] then showed that the isotopies with unit are geometrically and analytically significant because they imply corresponding compatible isotopies of the symplectic and Riemannian geometries, while analytically they permit the generalization of conventional hamiltonian mechanics in terms of the *isotopic time evolution*

$$dx/dt = [x, h] = x * h - h * x = xTh - hTx \quad (7)$$

which represents all conventional hamiltonian interactions, plus an additional class of generally nonlinear, nonlocal-integral and nonhamiltonian interactions.

For a second generalization of Lie and Lie-isotopic algebras into *Lie-admissible algebras* [5] whose product characterized the historical Hamilton's equations with external terms, we refer the interested reader for brevity to the appendices of ref. [4].

In conclusion, studies [3,4] have indicated that all generalizations of Lie algebras whose product characterizes the time evolution law are physically relevant. This reders significant the continued search of algebraic structures more general than the Lie-isotopic (and Lie-admissible) algebras, for the possible representation of yet more complex physical systems.

Along the lines, in this paper we shall introduce the *isotopic generalized structurable algebras*, or *isogeneralized structural algebras* for short, and show that they include the Lie algebras, as well as all their known axiom-preserving generalizations of graded, supersymmetric and isotopic type, as well as numerous additional associative and nonassociative algebras.

2. Isogeneralized structurable algebras

Lemma 1. *Let A be an associative algebra over a field Φ . Then A is a generalized structurable algebra equipped with the derivation*

$$D(x, y) := [L(x), L(y)] + [L(x), R(y)] + [R(x), R(y)].$$

Proof. From the fact that associative algebra is alternative algebra and so is a generalized structurable algebra equipped with the derivation $D(x, y)$, we can obtain the proof.

Theorem 2. *Let A be an associative algebra over Φ equipped with the above derivation $D(x, y)$. Then A (we denote it by $\hat{A}_1$) is a generalized structurable algebra over Φ with respect to the product and the derivation*

$$x * y := \alpha xy \quad \alpha \in \Phi$$

$$D^*(x, y) := D(x, y)$$

Theorem 3. *Let A be an associative algebra over Φ equipped with the above $D(x, y)$, let T be an nonsingular*

and Hermitean element and there is an element B such that $B^2 = T$ and $Bx, xB \in A$. Then A (we denote it by $\hat{A}_2$) is a generalized structurable algebra over Φ with respect to the product and the derivation defines by as follows;

$$a * b := aTb$$

$$D^*(a, b) := D(BaB, BbB).$$

Proof. It is clear that $D^*(x, y) = -D^*(y, x)$.
Since $D^*(a * b, c) = D(BaBBbB, BcB)$, we get

$$\sum_{(a,b,c)} D^*(a * b, c) = \sum_{(a,b,c)} D(BaBBbB, BcB).$$

We put $BaB = x, BbB = y$ and $BcB = z$.
From $\sum_{(x,y,z)} D(xy, z) = 0$, it follows that the identity

$$\sum_{(a,b,c)} D^*(a * b, c) = 0.$$

On the other hand,

$$\begin{aligned} D^*(a, b)(c * d) &= D(BaB, BbB)(cBBd) \\ &= (D(BaB, BbB)cBB)d + cBB(D(BaB, BbB)d) \\ &\text{(by the definition of derivation } D(x, y) \text{ of } A) \\ &= (D(BaB, BbB)c)BBd + cBB(D(BaB, BbB)d) \\ &\text{(by means of the associativity)} \end{aligned}$$

$$= (D^*(a, b)c) * d + c * (D^*(a, b)d)$$

This completes the proof.

Remark. If T is an Hermite matrix with elements Φ and $\Phi^{\frac{1}{2}} \subset \Phi$, then there exists a matrix B such that $B^2 = T$.

In fact, since $T = U^{-1} \begin{pmatrix} \alpha_1 & 0 \\ \cdot & \cdot \\ 0 & \alpha_n \end{pmatrix} U$, where U is unitary matrix,

putting $B := U^{-1} \begin{pmatrix} \sqrt{\alpha_1} & 0 \\ \cdot & \cdot \\ 0 & \sqrt{\alpha_n} \end{pmatrix} U$, we can obtain the identity $B^2 = T$.

Corollary. If T is a scalar matrix, and $\Phi^{\frac{1}{2}} \subset \Phi$, then the above theorem holds.

Proposition 4 Let A, T, B be as above Theorem 3.

Then A (we denote it by $\hat{L}_2$) is a generalized structurable algebra over Φ with respect to

$$[x, y] := xTy - yTx$$

$$D^*(x, y) := D(BxB, ByB.)$$

Theorem 5. Let A be an associative algebra equipped with $D(x, y)$ as Theorem 1 and w be an element such that $w^2 = w \neq 0, wA, Aw \in A$. Then $\hat{A}_3$ is a generalized

structurable algebra over Φ with respect to the product and the derivation as follows;

$$a * b := wawbw$$

$$D^*(a, b) := wD(waw, bw)w$$

Proof. It is clear that $D^*(a, b) = -D^*(b, a)$. Since $D^*(a * b, c) = wD(wawbw, cw)w$, we get

$$\begin{aligned} \sum_{(a,b,c)} D^*(a * b, c) &= \sum_{(a,b,c)} wD(wawbw, cw)w \\ &= \sum_{(a,b,c)} wD(wawbw, cw)w = 0. \end{aligned}$$

(by putting $waw = x, bw = y, cw = z$)
On the other hand, we obtain the following.

$$\begin{aligned} D^*(a, b)(c * d) &= (wD(waw, bw)w)(wcw)dw \\ &= wD(waw, bw)((wcw)(dw)) \\ &\text{(by means of the associativity and } w^2 = w) \\ &= w((D(waw, bw)wcw)dw) + \\ &\quad w((wcw)(D(waw, bw)dw)) \\ &= w(D^*(a, b)cw)(dw) + w(wcw)D^*(a, b)dw \\ &= wD^*(a, b)cw + wcwD^*(a, b)dw \\ &= (D^*(a, b)c) * d + c * (D^*(a, b)d). \end{aligned}$$

This completes the proof.

Proposition 6. *Let A, w be as above Theorem 5. Then $\hat{L}_3$ is a generalized structurable algebra over Φ with respect to*

$$[x, y] := wxwyw - wywxw$$

$$D^*(x, y) := wD(wxw, wyw)w$$

Theorem 7. *Let A be an associative algebra equipped with $D(x, y)$ as Theorem 1, let T is an element of nonsingular, Hermitian and there is an element B such that $B^2 = T$ and w be an element such that $w^2 = w$ and $BwawB \in A$ for all $a \in A$.*

Then $\hat{A}_4$ is a generalized structurable algebra over Φ with respect to the product and the derivation as follows;

$$a * b := wawTwbw$$

$$D^*(a, b) := wD(BwawB, BwbwB)w$$

Proof. It is clear that

$$D^*(a, b) = -D^*(b, a)$$

From the identity

$$D^*(a * b, c) = D^*(wawBBwbw, c)$$

$$= wD(BwawBBwbwB, BwcwB)w,$$

it follows that

$$\sum_{(a,b,c)} D^*(a * b, c) = 0$$

On the other hands,

$$\begin{aligned}
 D^*(a, b)(c * d) &= D^*(a, b)(wcwBBwdw) \\
 &= (wD(BwawB, BwbwB)w)(wcwBBwdw) \\
 &= wD(BwawB, BwbwB)(wcwBBwdw) \\
 &\quad (\text{by means of the associativity and } w^2 = w) \\
 &= w(D(BwawB, BwbwB)(wcwBB))(wdw) \\
 &\quad + w((wcwBB)(D(BwawB, BwbwB)wdw)) \\
 &= wD^*(a, b)cwBBwdw + w(wcwBB)D^*(a, b)dw \\
 &= (D^*(a, b)c) * d + c * (D^*(a, b)d)
 \end{aligned}$$

This completes the proof.

Proposition 8. Let A, T, B, w be as above Theorem 7. Then $\hat{L}_4$ is a generalized structurable algebra over Φ with respect to

$$[x, y] := wxwTwyw - wywTwxw$$

$$D^*(x, y) := wD(BwxwB, BwywB)w.$$

Remark. For $\hat{A}_5, \hat{L}_5$, the same results hold.

From the definition of a generalized structurable algebra, the following results is easily shown.

Proposition 9. *Let $(A, \circ)$ be a generalized structurable algebra equipped with $D(x, y)$. Then $(A, [,])$ is a generalized structurable algebra with equipped the same derivation $D(x, y)$ and with respect to the product $[x, y] := x \circ y - y \circ x$.*

Proposition 10. *Let $(A, \circ)$ be a generalized structurable algebra equipped with $D(x, y)$. Then $(A, \{ , \})$ is a generalized structurable algebra with equipped the same derivation $D(x, y)$ and with respect to the product $\{x, y\} := x \circ y + y \circ x$.*

Theorem 11. *Let A be a generalized structurable algebra over Φ equipped with $D(x, y)$ and T be an nonsingular and Hermite element and there is an element B such that $B^2 = T$, BxB are associative .*

Then A (we denote it by $\hat{A}_2$) is a generalized structurable algebra over Φ with respect to

$$a * b := aTb$$

$$D^*(a, b) := D(BaB, BbB)$$

Proof. The proof is same as Theorem 2.

Remark *Assume that wxw and $wxwyw$ are associative respectively, then we have same results as Theorem 5, so Proposition 6.*

Remark *Assume that $wxwTwyw$, $BwxwB$ are associative respectively, we have same result as Theorem 7 , so Proposition 8.*

3. isotopies of generalized structurable superalgebra

As a generalization of a generalized structurable algebra, we may define a generalized structurable superalgebra over

a field Φ of $\text{ch } \Phi \neq 2 \text{ or } 3$ to be a nonassociative superalgebra A equipped with a superderivation $D(x, y)$ in which the following conditions are satisfied

$$D(x, y) = -(-1)^{\deg x \deg y} D(y, x)$$

$$(-1)^{\deg x \deg z} D(xy, z) + (-1)^{\deg y \deg x} D(yz, x) +$$

$$(-1)^{\deg z \deg y} D(zx, y) = 0$$

for all x, y, z in A .

Remark. We note that the definition of superalgebra and superderivation imply the following; let A be an algebra and M be an abelian group, then M -grading of A is a decomposition of A into a direct sum of subspace $A = \bigoplus_{\alpha \in M} A_{\alpha}$ for which $A_{\alpha} A_{\beta} \subset A_{\alpha+\beta}$. An algebra A equipped with an M -grading is said to be superalgebra, and if $a \in A_{\alpha}$, then we denote by $\deg a = \alpha$.

It is said to be a superderivation of degree s if an endomorphism $D \in \text{End } A$ satisfies the property $D(xy) = D(x)y + (-1)^{s \deg x} xD(y)$.

Proposition 12. Let A be a Lie superalgebra. Then A is a generalized structurable superalgebra equipped with the superderivation

$$D(x, y) = ad[x, y]$$

Proof. The proof is clear and so omit it.

Theorem 13. Let A be a generalized structurable superalgebra equipped with the superderivation $D(x, y)$. Then

$(A, [,]) is a generalaized structurable superalgebra with equipped the same superderivation $D(x, y)$ and with respect to the product $[x, y] := xy - (-1)^{deg x deg y} yx$.$

Proof. (the proof of superderivation)

$$\begin{aligned}
 & D(x, y)[z, w] \\
 &= D(x, y)(zw - (-1)^{deg z deg w} wz) \\
 &= D(x, y)(zw) - (-1)^{deg z deg w} D(x, y)(wz) \\
 &= (D(x, y)z)w + (-1)^{deg z(deg x + deg y)} z(D(x, y)w) \\
 &\quad - (-1)^{deg z deg w} (D(x, y)w)z - \\
 &\quad (-1)^{deg z deg w} (-1)^{deg w(deg x + deg y)} w(D(x, y)z) \\
 &= [D(x, y)z, w] + (-1)^{deg z(deg x + deg y)} [z, D(x, y)w].
 \end{aligned}$$

(the proof of super generalized Jacobi identity)
On the other hand, we have the following

$$\begin{aligned}
 & \sum_{(x, y, z)} (-1)^{deg x deg z} D([x, y], z) \\
 &= \sum_{(x, y, z)} (-1)^{deg x deg z} D(xy - (-1)^{deg x deg y} yx, z) \\
 &= \sum_{(x, y, z)} (-1)^{deg x deg z} D(xy, z) -
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{(x,y,z)} (-1)^{\deg x(\deg y + \deg z)} D(yx, z) \\
 &= \sum_{(x,y,z)} (-1)^{\deg x \deg z} D(xy, z) \\
 & - (-1)^{\deg x \deg y + \deg x \deg z + \deg y \deg z} \sum_{(x,y,z)} (-1)^{\deg y \deg z} D(yx, z) \\
 &= 0.
 \end{aligned}$$

This completes the proof.

4. Concluding remarks

In this paper we have shown that:

- a) the generalized structurable algebras as per axioms (2) contains all Lie algebras over a field of characteristic different than 2 or 3, as well as all their physically relevant, graded and supersymmetric extensions;
- b) the idogeneralized structural algebras contain the additional class of isotopic Lie, graded and supersymmetric extensions, thus including all algebras of Lie type that are known to be physically significant at this writing; and that
- c) the isogeneralized structurable algebras contain additional associative and nonassociative algebras of possible mathematical and physical interest we hope to investigate in subsequent studies.

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ON THE INVERSE PROBLEM IN THE HIGHER-ORDER ANALYTICAL MECHANICS

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Abstract

The inverse problem and universality theorems in Newtonian Mechanics have been extensively studied by R.M. Santilli, [11], [12]. In Analytical Mechanics or in the Higher-order Lagrangian Mechanics these problems were solved by M. Crampin [2], D. Krupka [5], M. de Leon [6] and many others [8]. Generally, these studies are focused on the Euler-Lagrange equations. But, besides of the mentioned equations there exists also other important equations determined only by a Lagrangian of order $k \geq 1$. For instance the integral curves of the canonical k -spray are given by a system of differential equations which depend on the Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ and has a geometrical meaning. Therefore in the higher-order Analytical Mechanics the inverse problem must be accordingly formulated.

In this lecture we present the notion of k -osculator bundle, k -spray, h -paths and variational problem for the Lagrangians of order k , $k \in N^*$. we formulate the inverse problem for each one of the systems $\overset{0}{E}_i(L) = \dots = \overset{k-1}{E}_i(L) = 0$ and prove that the first one, $\overset{0}{E}_i(L) = 0$ is self-adjoint, while the others $\overset{1}{E}_i(L) = \dots = \overset{k-1}{E}_i(L) = 0$ are non self-adjoint. Introducing the energies of order k and $k - 1$ we discuss the laws of conservation and the Hamilton-Jacobi equations.

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1. Let M be a real n -dimensional manifold and $(\text{Osc}^k M, \pi, M)$ its k -osculator bundle. We briefly describe the manifold $E = \text{Osc}^k M$. The curves in M , $\delta, \sigma : I \rightarrow M$, which have in common a point $x_0 \in M$, $x_0 = \delta(0) = \sigma(0)$, ($0 \in I$), have in x_0 a "contact of order k ", ($k \in \mathbb{N}^*$) if for any function $f \in \mathcal{F}(U)$, $x_0 \in U$, U being an open set in M , we have

$$(1.1) \quad \left. \frac{d^\alpha}{dt^\alpha} (f \circ \rho)(t) \right|_{t=0} = \left. \frac{d^\alpha}{dt^\alpha} (f \circ \sigma)(t) \right|_{t=0}, \quad (\alpha = 1, \dots, k).$$

The relation "contact of order k " is a relation of equivalence. We denote by $[\rho]_{x_0}$ a class of equivalence. It will be called a " k -osculator space" in the point x_0 to the manifold M . Let $\text{Osc}_{x_0}^k$ be the set of k -osculator spaces in x_0 to M and let us consider the set

$$\text{Osc}^k M = \bigcup_{x_0 \in M} \text{Osc}_{x_0}^k$$

and the mapping $\pi : ([\rho]_{x_0}) \in \text{Osc}^k M \rightarrow x_0 \in M$.

The set $E = \text{Osc}^k M$ has a natural differentiable structure induced by that of M such that π becomes a differentiable surjection. Indeed, let (U, φ) be a local chart of M , $x_0 \in U$ and $\rho : I \rightarrow M$ a smooth curve represented in (U, φ) by

$$x^i = x^i(t), \quad t \in I, \quad x_0^i = x^i(0), \quad (0 \in I),$$

x_0^i being the coordinates of the point $x_0 \in U$ in (U, φ) .

The class $[\rho]_{x_0}$ has a representative element given by

$$x^{*i}(t) = x^i(0) + t \frac{dx^i}{dt}(0) + \dots + t^k \frac{1}{k!} \frac{d^k x^i}{dt^k}(0), \quad t \in (-\varepsilon, \varepsilon) \subset I.$$

The previous polynomial functions are defined by the coefficients

$$(1.2) \quad x_0^i = x^i(0), \quad y_0^{(1)i} = \frac{dx^i}{dt}(0), \dots, y_0^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}(0).$$

Therefore, the pair $(\pi^{-1}(U), \Phi)$, with

$$\Phi : \pi^{-1}(U) \subset E \rightarrow \varphi(U) \times R^{kn} \subset R^{(k+1)n}, \quad \Phi([\rho]_{x_0}) = (x_0^i, y_0^{(1)i}, \dots, y_0^{(k)i})$$

is a local chart on E , ($i, j, \dots = 1, \dots, n$).

So that a differentiable atlas $\mathcal{A}_M$ on M determines a differentiable atlas $\mathcal{A}_E$ on E . The triple $(\text{Osc}^k M, \pi, M)$ is a differentiable bundle.

Using (1.2) we obtain the transformations of the local coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i}) \longrightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, \dots, \tilde{y}^{(k)i})$ in the form

$$\begin{aligned}
 \tilde{x}^i &= \tilde{x}^i(x^1, \dots, x^n), \det \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| \neq 0, \\
 \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\
 (1.3) \quad 2\tilde{y}^{(2)i} &= \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}, \\
 \dots \dots \dots \\
 k y^{(k)j} &= \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \dots + k \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j},
 \end{aligned}$$

where we have

$$\frac{\partial \tilde{x}^i}{\partial x^j} = \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} = \dots = \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k)j}}, \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(2)i}}{\partial y^{(1)j}} = \dots = \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k-1)j}} \text{ etc.}$$

2. On the differentiable manifold $E = \text{Osc}^k M$ there exist the k -vertical distribution $V_1, \dots, V_k$ with the properties: V_1 is the vertical distribution on E , the distribution V_2 is included in the distribution V_1 and $\dim V_1 - \dim V_2 = n$, and so on, we have $V_k \subset V_{k-1}$ and $\dim V_{k-1} - \dim V_k = n$. These distributions are integrable. On E there exist the k -Liouville vector fields given by

$$\begin{aligned}
 (2.1) \quad \overset{1}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(k)i}}, \overset{2}{\Gamma} = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}}, \dots, \\
 \overset{k}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \dots + k y^{(k)i} \frac{\partial}{\partial y^{(k)i}}.
 \end{aligned}$$

Also, on E there exists a k -tangent structure $J : \mathcal{X}(E) \longrightarrow \mathcal{X}(E)$ such that

$$\begin{aligned}
 (2.2) \quad J \left(\frac{\partial}{\partial x^i} \right) &= \frac{\partial}{\partial y^{(1)i}}, J \left(\frac{\partial}{\partial y^{(1)i}} \right) = \frac{\partial}{\partial y^{(2)i}}, \dots, \\
 J \left(\frac{\partial}{\partial y^{(k-1)i}} \right) &= \frac{\partial}{\partial y^{(k)i}}, J \left(\frac{\partial}{\partial y^{(k)i}} \right) = 0.
 \end{aligned}$$

Clearly, we have

$$(2.3) \quad J(\overset{1}{\Gamma}) = 0, J(\overset{2}{\Gamma}) = 0, \dots, J(\overset{k}{\Gamma}) = \left(\overset{k-1}{\Gamma} \right)$$

and J is an integrable structure.

A k -spray on E is a vector field $S \in \mathcal{X}(E)$ with the property:

$$(2.4) \quad JS = \frac{k}{1}.$$

In the local coordinates a k -spray S can be expressed by

$$(2.5) \quad S = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + k y^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (1+k) G^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}.$$

The functions $G^i(x, y^{(1)}, \dots, y^{(k)})$ are called the coefficients of the k -spray S .

The existence of a k -spray S is essential in the geometry of the higher order Lagrange space [7], [10]. Indeed, S determines completely a nonlinear connection on E with the dual coefficients given by

$$\begin{aligned} M_{(1)}^i{}_{j} &= \frac{\partial G^i}{\partial y^{(k)i}}, M_{(2)}^i{}_{j} = \frac{1}{2} \left(S M_{(1)}^i{}_{j} + M_{(1)}^i{}_{\tau} M_{(1)}^{\tau}{}_{j} \right), \dots, \\ M_{(k)}^i{}_{j} &= \frac{1}{k} \left(S M_{(k-1)}^i{}_{j} + M_{(1)}^i{}_{\tau} M_{(k-1)}^{\tau}{}_{j} \right). \end{aligned}$$

These coefficients depend on the k -spray S only.

The integral curves of the k -spray S from (2.5) are the solutions of the following system of differential equations

$$(2.6) \quad \begin{aligned} y^{(1)i} &= \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}, \\ \frac{d^{k+1} x^i}{dt^{k+1}} + (k+1) G^i \left(x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k} \right) &= 0. \end{aligned}$$

The converse property holds, too.

More general, a h -path on the differentiable manifold M is a solution curve of a system of ordinary differential equations on M , defined on every domain of local chart by

$$(2.7) \quad \frac{d^{h+1} x^i}{dt^{h+1}} + (h+1) G^i \left(x, \frac{dx}{dt}, \dots, \frac{1}{h!} \frac{d^h x^i}{dt^h} \right) = 0.$$

which is preserved by the transformation of coordinates (1.3).

For $h = k$ the equation (2.7) defines a k -spray S on the manifold E .

We shall see that the extremal curves of the integral of action of a higher order regular Lagrangian L are the $2k - 1$ paths on M .

3. A differentiable Lagrangian of order k , ($k \in N^*$) is a mapping $L: E \rightarrow R$ of class C in the points $(x, y^{(1)}, \dots, y^{(k)})$ of E where $y^{(1)} \neq 0$ and which is continuous in all points of E . The Lagrangian L is called regular if the Hessian with elements

$$(3.1) \quad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}$$

is nonsingular.

The quantity g_{ij} from (3.1) is a distinguished tensor field (shortly, d -tensor) named the fundamental tensor field of the Lagrangian L .

Let $c: t \in [0, 1] \rightarrow (x^i(t)) \in M$ be a smooth curve in M . The integral of action of the differentiable Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ is defined by

$$(3.2) \quad I(c) = \int_0^1 L \left(x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) dt.$$

Applying the variational principle to $I(c)$ we deduce:

Theorem 3.1. *In order that the curve c be an extremal curve of the functional $I(c)$ it is necessary that the following Euler-Lagrange equations hold:*

$$(3.3) \quad \begin{aligned} {}^0 E_i(L) &= \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \dots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} = 0, \\ y^{(1)i} &= \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}. \end{aligned}$$

Theorem 3.2. *For any differentiable Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ the quantities ${}^0 E_i(L)$ determine a d -covector field.*

Now we can prove:

Lemma 3.1. *For any differentiable Lagrangian and any differentiable function $\phi(t)$ we have along of a smooth curve $c: t \in I \rightarrow c(t) \in M$,*

$$(3.4) \quad {}^0 E_i(\phi L) = \phi {}^0 E_i(L) + \frac{d\phi}{dt} \frac{1}{k!} E_i(L) + \dots + \frac{d^k \phi}{dt^k} \frac{k}{k!} E_i(L)$$

where

$$\begin{aligned}
 {}^1E_i(L) &= \sum_{\alpha=1}^k (-1)^\alpha \frac{1}{\alpha!} \binom{\alpha}{\alpha-1} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \frac{\partial L}{\partial y^{(\alpha)}_i}, \\
 {}^2E_i(L) &= \sum_{\alpha=2}^k (-1)^\alpha \frac{1}{\alpha!} \binom{\alpha}{\alpha-2} \frac{d^{\alpha-2}}{dt^{\alpha-2}} \frac{\partial L}{\partial y^{(\alpha)}_i}, \\
 &\dots\dots\dots \\
 {}^kE_i(L) &= (-1)^k \frac{1}{k!} \frac{\partial L}{\partial y^{(k)}_i}.
 \end{aligned}
 \tag{3.5}$$

Theorem 3.3. *For any differentiable Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ every one of quantities ${}^1E_i(L), \dots, {}^kE_i(L)$ is a d -covector field.*

These d -covectors were discovered by Craig and Synge [7], [10].
Now we can prove, without difficulties:

Theorem 3.4. *If $L(x, y^{(1)}, \dots, y^{(k)})$ is a regular Lagrangian, then the equations: ${}^0E_i(L) = 0$ give $2k - 1$ paths; ${}^1E_i(L) = 0$ give $2k - 2$ paths and so on, ${}^{k-1}E_i(L) = 0$ give k paths.*

Indeed, any Craig-Synge equation

$${}^0E_i(L) = 0, \dots, {}^{k-1}E_i(L) = 0
 \tag{3.6}$$

is of the form

$$g_{ij} \frac{d^{h+1}x^j}{dt^{h+1}} + G_i \left(x, \dots, \frac{d^h x}{dt^h} \right) = 0, \quad h = k, \dots, 2k - 1.
 \tag{3.7}$$

It has a geometrical meaning and $\text{rank} \|g_{ij}\| = n$.

Consequently, the inverse problem concerning the existence of the Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ with the property that the equations of the form (3.7) be identically to one of the equations (3.6) can be formulated.

The solution of this problem for $k = 1$ in the case of Newtonian systems was given by R.M. Santilli [11], [12]. In the general case $h = 2k - 1$ the problem was solved by M. Crampin [2], D. Krupka [5] and many others.

So (cf. Santilli [11], pag. 196), we have:

Theorem 3.5. *The system of differential equations (3.7), $h = 2k - 1$ being given, there exists the Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ with the property that this system can be written in the form of the Euler-Lagrange equations $\overset{0}{E}_i(L) = 0$ if and only if the system (3.7), $h = 2k - 1$ is self-adjoint.*

But, for $h = k, \dots, 2k - 2$, $k > 1$ every one of the systems (3.7) is non self-adjoint.

Theorem 3.6. *Each differential equation of the system $\overset{1}{E}_i(L) = 0$, $\dots$, $\overset{k-1}{E}_i(L) = 0$ is non self-adjoint, for $k = 2, 3, \dots, N$.*

Proof. Each one of these systems of equations is of the form (3.7) with $h = k, \dots, 2k - 2$. Denoting by

$$F_i(x, y^{(1)}, \dots, y^{(h+1)}), \left\{ y^{(1)} = \frac{dx^i}{dt}, \dots, y^{(h+1)} = \frac{1}{(h+1)!} \frac{d^{h+1}x^i}{dt^{h+1}} \right\}$$

the left side of the equations (3.7) we can write it in the form

$$(3.7)' \quad F_i(x, y^{(1)}, \dots, y^{(h+1)}) = 0.$$

If h is an even natural number one of the Helmholtz conditions (cf. Santilli [11]) is given by

$$\frac{\partial F_i}{\partial y^{(h+1)m}} + \frac{\partial F_m}{\partial y^{(h+1)i}} = 0.$$

This condition is equivalent to $g_{ij} = 0$ via (3.7). But this is an impossibility, since $\text{rank } \|g_{ij}\| = n$, and this condition has a geometrical meaning, even all Helmholtz conditions have not a geometrical character. q.e.d.

4. For $k = 1$, the Euler-Lagrange equations $\overset{0}{E}_i(L) = 0$ are of the second order. So, we can transform them, via Legendre transformation, in the Hamilton-Jacobi system of differential equations, which is of the first order. In his books [11], [12], R.M. Santilli made a good theory on the property of universality of these systems.

Santilli's theory can be extended to the higher order Lagrangians.

Following the Manuel de Leon's and P. Rodrigues book [6], we determine the Hamilton-Jacobi equations for the higher order Lagrangians.

Let us consider the main invariants (cf. Miron, [7]) of the Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$:

$$(4.1) \quad \overset{1}{I}(L) = \mathcal{L}_1 L, \dots, \overset{k}{I}(L) = \mathcal{L}_k L$$

where $\mathcal{L}$ is the Lie derivative.

Along of a smooth curve $c : I \longrightarrow M$, we can determine the energy of order k of the Lagrangian L , in the form:

$$(4.2) \quad \mathcal{E}_c^k(L) = \overset{k}{I}(L) - \frac{1}{2} \frac{d}{dt} \overset{k-1}{I}(L) + \dots + (-1)^k \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \overset{1}{I}(L) - L.$$

There are known the following theorem, [6], [7]:

Theorem 4.1. *For any differentiable Lagrangian of order k , L , and any curve $c : I \longrightarrow M$, the variation of the energy $\mathcal{E}_c^k(L)$ along of c , is given by*

$$(4.3) \quad \frac{d\mathcal{E}_c^k(L)}{dt} = - \overset{0}{E}_i(L) \frac{dx^i}{dt}.$$

Theorem 4.2. *The energy of order k , $\mathcal{E}_c^k(L)$, is conserved along of every solution curve of the Euler-Lagrange equations $\overset{0}{E}_i(L) = 0$.*

Writting $\mathcal{E}_c^k(L)$ in the form

$$(4.4) \quad \mathcal{E}_c^k(L) = p_{(1)i} \frac{dx^i}{dt} + \dots + p_{(k)i} \frac{d^k x^i}{dt^k} - L$$

we obtain the Jacobi-Ostrogradski momenta:

$$(4.5) \quad \begin{aligned} p_{(1)i} &= \frac{\partial L}{\partial y^{(1)i}} - \frac{1}{2!} \frac{d}{dt} \frac{\partial L}{\partial y^{(2)i}} + \dots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial L}{\partial y^{(k)i}}, \\ p_{(2)i} &= \frac{1}{2!} \frac{\partial L}{\partial y^{(2)i}} - \frac{1}{3!} \frac{d}{dt} \frac{\partial L}{\partial y^{(3)i}} + \dots + (-1)^{k-2} \frac{1}{k!} \frac{d^{k-2}}{dt^{k-2}} \frac{\partial L}{\partial y^{(k)i}}, \\ &\dots\dots\dots \\ p_{(k)i} &= \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}}. \end{aligned}$$

It is not difficult to prove that these momenta transform with respect to (1.3) as follows:

$$(4.6) \quad p_{(1)i} = \frac{\partial \tilde{y}^{(1)m}}{\partial y_{(1)i}} \tilde{p}_{(1)m} + \cdots + \frac{\partial \tilde{y}^{(k)m}}{\partial y_{(1)i}} \tilde{p}_{(k)m}, \dots, p_{(k)i} = \frac{\partial \tilde{y}^{(k)m}}{\partial y_{(k)i}} \tilde{p}_{(k)m}.$$

Lemma 4.1. *The following formulae hold:*

$$(4.7) \quad \begin{aligned} \frac{dp_{(1)i}}{dt} &= \frac{\partial L}{\partial x^i} - \overset{0}{E}_i(L), \\ \frac{dp_{(2)i}}{dt} &= \frac{\partial L}{\partial y^{(1)i}} - p_{(1)i}, \\ \dots\dots\dots \\ \frac{dp_{(k)i}}{dt} &= \frac{1}{(k-1)!} \frac{\partial L}{\partial y^{(k-1)i}} - p_{(k-1)i}. \end{aligned}$$

From which we deduce:

Theorem 4.3. *Along of every solution curve of the Euler-Lagrange equation $\overset{0}{E}_i(L) = 0$ the following Hamilton-Jacobi equations hold:*

$$(4.8) \quad \begin{aligned} \frac{d\mathcal{E}_c^k(L)}{dp_{(\alpha)i}} &= \frac{d^\alpha x^i}{dt^\alpha}, \quad (\alpha = 1, \dots, k) \\ \frac{d\mathcal{E}_c^k(L)}{\partial x^i} &= -\frac{dp_{(1)i}}{dt}, \\ \frac{d\mathcal{E}_c^k(L)}{\partial y^{(\alpha)i}} &= -\alpha! \frac{dp_{(\alpha+1)i}}{dt}, \quad (\alpha = 1, \dots, k). \end{aligned}$$

Remarks. 1° Using the Hamilton-Jacobi equations (4.8) Santilli's theorems of universality, valid for $k = 1$, can be extended to general case.

2° A theory of fields, based on the Hamilton-Jacobi equations (4.8) can be find in the book of M. de Leon and P. Rodrigues, [6].

5. In the paper [7], it is introduced the notion of energy of order $k - 1$, $\mathcal{E}_c^{k-1}(L)$ in order to prove the Nöther type theorem for the differentiable Lagrangians $L(x, y^{(1)}, \dots, y^{(k)})$. It is given by

$$(5.1) \quad \mathcal{E}_c^{k-1}(L) = -\frac{1}{2} \overset{k-1}{I}(L) + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-2}}{dt^{k-2}} \overset{1}{I}(L).$$

Taking into account the expressions (4.2) and (4.4) of the energy $\mathcal{E}_c^{k-1}(L)$ we deduce, without difficulties:

Theorem 5.1. *The energy of order $k - 1$, along of a smooth curve $c : I \longrightarrow M$ has the following property*

$$(5.2) \quad \frac{d\mathcal{E}_c^{k-1}(L)}{dt} = p_{(1)i} \frac{dx^i}{dt} + \cdots + p_{(k)i} \frac{d^k x^i}{dt^k} - \overset{k}{I}(L).$$

Theorem 5.2. *The energy of order $k - 1$, $\mathcal{E}_c^{k-1}(L)$, is conserved along of a smooth curve $c : I \longrightarrow M$ if and only if along of c , we have*

$$(5.3) \quad \mathcal{L}_{\Gamma}^k(L) = p_{(1)i} \frac{dx^i}{dt} + \cdots + p_{(k)i} \frac{d^k x^i}{dt^k}.$$

Finally, we remark that along of a smooth curve $c : I \longrightarrow M$, the main invariant $\overset{k}{I}(L)$ has the expression

$$\overset{k}{I}(L) = \frac{\partial L}{\partial y^{(1)i}} \frac{dx^i}{dt} + \cdots + \frac{k}{k!} \frac{\partial L}{\partial y^{(k)i}} \frac{d^k x^i}{dt^k}.$$

Therefore, we can write (5.1) in the form

$$\frac{d\mathcal{E}_c^{k-1}(L)}{dt} = \left(p_{(1)i} - \frac{\partial L}{\partial y^{(1)i}} \right) \frac{dx^i}{dt} + \cdots + \left(p_{(k)i} - \frac{k}{k!} \frac{\partial L}{\partial y^{(k)i}} \right) \frac{d^k x^i}{dt^k},$$

which allows us to express theorem 5.2 in a new corresponding form.

Remarks. These considerations on the energies of order k and $k - 1$ can be extended to the another energies of order $k - 2, \dots, 1$ and used in an attempt to study the Santilli Hadronic Mechanics to higher order analytical mechanics, [4], [13].

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